10 JANUARY 2012

MA 375

Symbolics

→ Numbers : naturals $\mathbb{N}$ → logical symbols : $\in$ "is element of" ($n \in \mathbb{N}$)

integers $\mathbb{Z}$: $\notin$ "is not an element of"

rationals $\mathbb{Q}$: $\Longrightarrow$ "implies" ($a<1 \Rightarrow a<2$)

real $\mathbb{R}$: $\exists$ "exists"

complex $\mathbb{C}$: $\forall$ "for all"

→ mathematical sentences are rendered w/ much simplicity

ex $\left[a = \lim_{n\to\infty} a_n\right] \Rightarrow \left[\forall\ \varepsilon > 0\ \ \exists\ n \in \mathbb{N},\ n \gg 0 : |a_i - a| < \varepsilon\ \forall\ i > n\right]$

INDUCTION

→ used to prove infinitely many statements at one.

→ suppose $P(n)$ is a statement involving parameter $n \in \mathbb{N}$

ex $P(n)$ = "the sum of the first n natural number equals $\frac{n(n+1)}{2}$

We want to prove $P(n)$ for $\forall\ n \in \mathbb{N}$

$\Rightarrow$ how do we avoid the need of proving ∞ statements?

Basic Idea

① Check $P(n)$ explicitly for some (usually just one) small n.

② Relate the truth of the statement $P(n)$ to the truth of the statement $P(m)$ for some (or several) values that proceed n, $m<n$

ex

$$P(n) = \sum_{i=0}^{i=n} i = \frac{n(n+1)}{2} \quad \text{for } n = 0, 1, 2, \ldots$$

1) Base Case: for $n=0$

LHS: $\sum i = 0$ RHS: $\frac{0(0+1)}{2} = 0$

LHS = RHS: for $P(0)$, $P(n)$ is true.

2) Inductive Step: need to relate $P(n)$ to $P(m)$ where $m<n$

it is useful to consider $m = n-1$ (this is not always the case)

Let $P(n) = \sum_{i=0}^{n} i = \frac{n(n+1)}{n}$

$P(n-1) = \sum_{i=0}^{n} i = \frac{n-1(n)}{n}$

: if I can relate given 2 statements, and since I've proved an instance of the lower statement, the whole truth can be discovered.

Method:

a. $P(n) = 1 + 2 + \dots + n = \frac{n(n+1)}{2}$

b. $P(n-1) = 1 + 2 + \dots \; n-1 = \frac{(n-1)n}{2}$

Consider $P(n) - P(n-1)$. Then,

$$n = \frac{n(n+1)}{2} - \frac{n(-1)n}{2} = \frac{n^2 + n - n^2 + n}{2} = \frac{2n}{2} = n \quad \checkmark$$

If I add n to statement b, which was proven in the base case, then adding n to both LHS & RHS will result in statement a, which must be true.

We conclude:

If $P(n-1)$ is true $\left(\sum_{i=0}^{n-1} i = \frac{n(n-1)}{2}\right)$, then adding "n" on both side we find $\sum_{n=0}^{n} i = \frac{n(n+1)}{2}$ so that $P(n)$ is also true.

NOTE: Both base step & inductive step is needed

ex

$$P(n) = \sum_{i=0}^{n} i = \frac{n(n+1)}{2} + 6.$$

→ Base case, of course, fails

HOWEVER, INDUCTIVE STEP WILL STILL HOLD. The inductive step simply related $P(n)$ and $P(n-1)$ so that if $P(n-1)$ is true, $P(n)$ must also be true.

MORAL: logical step does not rectify contradictory axiom. with shit you can only make logical shits.

Remark: $n-1$ case/consideration is not always adapt.

INFINITE ~~DESCENT~~

	PURPOSE	METHOD
INDUCTION	$P(n)$ is true for all n	check $n=0$ or small n; reduce $P(n)$ to $P(n-1)$ or $P(m)$, $m<n$
INFINITE DESCENT	$P(n)$ is false $\forall n$	check $P(0)$ = False; pretend that $P(n)$ is true and show if $P(m)$ is also true for $m<n$

Ex. $\sqrt{2}$ is not rational

→ we want to show that $\sqrt{2}$ is not a quotient of two rational numbers

↳ Let $P(n)$ = "there is a natural # m such that $\frac{m}{n} = \sqrt{2}$"

↳ we like to show that $P(n)$ is false for all n

↳ $P(0)$ is false because $\frac{m}{0}$ is undefined (this is enough, I think)

$P(1)$ is false because $\frac{m}{1} = m$ ~~is n~~ cannot be $\sqrt{2}$ if $m \in \mathbb{N}$

= if we can find a crank that would inevitably prove $P(0)$ or $P(1)$ to be true...

Suppose $P(n)$ is true.

$$\sqrt{2} = \frac{m}{n} \text{ for some } m \in \mathbb{N}$$

- then, $\left(\frac{m}{n}\right)^2 = 2$. or $2n^2 = m^2$
- we see that m^2 is even, and therefore $2m' = m$ where $m' \in \mathbb{N}$
- we rewrite $2n^2 = m^2 = 4m'^2$; we not that

$$n^2 = 2m'^2 \text{ and } n \text{ is even.}$$

- $2n' = n^2$ where $n' \in \mathbb{N}$.
- So $\frac{m}{n} = \frac{2m'}{2n'} = \frac{m'}{n'} = \frac{m''}{n''} = \dots \frac{m^*}{n^*}$.

↳ If you blieve in $P(n)$, you must also believe in $P(\frac{n}{2})$

$P(n) \Rightarrow P(\frac{n}{2})$ is the descent.

↳ as n reduces by half, it would inevitably lead to conclusion that $P(0)$ or $P(1)$ is true, which we have shown that it is not. Therefore, our pretending of $P(n)$ is true is false, and $P(n)$ is false for $\forall n$.

IMPORTANT PROPERTY THAT MAKES INDUCTION & DESCENT TICK:

If you take any collection of natural numbers with at least 1 member, then this collection must have a smallest element.

→ This is true because $\mathbb{N}$ is well orderded.

→ Induct/Descent will not work if the index set were integers because there is no lowest step for $\mathbb{Z}$ (or $\mathbb{R}$, or $\mathbb{C}$) or $[0,1]$. By step, I mean that if I tak $(0,1)$ collection, we cannot define a lowest number.