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1 Problem 2-b

Assuming 2a. Assuming $q > 2$. If $q = 2$, this is trivially true.

Let $\{p_1, p_2, \ldots, p_n\}$ be all the primes of the form $qb + 1$. Let $\alpha = \Pi p_i$. Observe that $\alpha = Mq + 1$ for some M.

Let p be a prime that divides $f_q(\alpha)$.

$$f_q(\alpha) = \frac{(\Pi p_i)^q - 1}{\alpha - 1}$$

Since no p_i divides $(\Pi p_i)^q - 1$, the prime p must be q. Hence, this implies $f_q(\alpha) = q^m$ for some m. Then

$$\alpha^q - 1 = q^m(Mq - 1) \Rightarrow (Mq + 1)^q - 1 = q^m(Mq + 1 - 1)$$

$$(Mq)^q + q(Mq)^{q-1} + \binom{q}{2}(Mq)^{q-2} + \ldots + \binom{q}{2}(Mq)^2 + qMq + 1 - 1 = q^m Mq$$

$$(Mq)^q + q(Mq)^{q-1} + \binom{q}{2}(Mq)^{q-2} + \ldots + \binom{q}{2}(Mq)^2 + qMq = q^m Mq$$

$$(Mq)^{q-1} + q(Mq)^{q-2} + \binom{q}{2}(Mq)^{q-3} + \ldots + \binom{q}{2}(Mq)^2 + q = q^m$$

$$qK + 1 = q^{m-1}$$

(every term in the LHS except for q has at least a factor of q^2 since $q > 2$.) This implies that $m = 1$. Hence $f_q(\Pi p_i) = q$. But we can let $\alpha = p_1^2 p_2 \ldots p_n$ and the same argument holds, this gives the contradiction. QED.