1. Find the limit function of the sequence x^n on $[0,1]$ and show that we have uniform convergence on any compact subset of $[0,1)$.

2. Let f_n be continuous and real-valued on a compact set K. If $f_n \to f$ uniformly, show f is also continuous.

Switching Limits and integrals

3. Suppose $f_n \in \mathfrak{R}[0,1]$ with $\int_0^1 f_n = 1$ for all $n = 1, 2, ...$ and that f_n converges pointwise to say f (i.e. $f_n \to f$.)
   (a) Prove or disprove $f \in \mathfrak{R}[0,1]$.
   (b) What if we also suppose f_n are bounded (uniformly in n and x)?
   (c) Now suppose $f \in \mathfrak{R}[0,1]$ also. Is it true that $\int f_n \to \int f$?

Root and Ratio tests

4. Let $a_n \in \mathbb{R}$ or $\mathbb{C}$ and $f(x) = \sum_{n=0}^{\infty} a_n x^n$ wherever defined. Let $f_N(x) = \sum_{n=0}^{N} a_n x^n$.
   (a) Show that the series converges absolutely whenever
   $$|x| < R := (\limsup |\frac{a_{n+1}}{a_n}|)^{-1}$$
   .
   (b) Show that the f_N converge absolutely and uniformly on any compact subset K of $N_R(0)$. (Hint: If $x \in K$ there is some $0 < r < R$ such that $|x| < r$. Repeat the proof of the ratio-test, i.e. bound by a geometric series)
   (c) Now define $R' := (\limsup \sqrt[n]{|a_n|})^{-1}$ and show the same two results as above.
   (d) Even better, show that if $|x| > R'$ our series diverges.
   Corollary : $R' \leq R$
   (e) Can we find a sequence a_n where $R' < R$?
   (f) Show that if $\lim |\frac{a_{n+1}}{a_n}| = R^{-1}$ exists, then $R = R'$.