12 April 2002 **Name:**_____

EE 438 **Exam No. 3** **Spring 2002**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are permitted.

1. (30 pts.)

   a. A wide-sense stationary process $X[n]$ with mean 1 and autocorrelation function

$$r_{XX}[n] = \begin{cases} 1, & n = 0 \\ 0.5, & n = \pm 1 \\ 0, & \text{else} \end{cases}$$

   is input to the system described by $y[n] = x[n] - x[n-1]$.

   Find the mean $\mu_Y[n]$ and autocorrelation $r_{YY}[n]$ of the output process $Y[n]$.

   b. The same wide-sense stationary process $X[n]$ as in part a. above is input to the system described by $y[n] = x[2n]$.

   Find the mean $\mu_Y[n]$ and autocorrelation $r_{YY}[n]$ of the output process $Y[n]$.

   c. The same wide-sense stationary process $X[n]$ as in parts a. and b. above is input to the system described by

$$y[n] = \begin{cases} x[n/2], & n/2 \text{ is an integer} \\ 0, & \text{else} \end{cases}$$

   Find the mean $\mu_Y[n]$ and autocorrelation $r_{YY}[n]$ of the output process $Y[n]$.

a. $\mu_Y[n] = E[Y[n]]$

$= E[X[n] - X[n-1]]$

$= E[X[n]] - E[X[n-1]]$

$= 1 - 1$

$= 0$

$r_{YY}[n] = E[Y[m]Y[m+n]]$

$= E[(X[m] - X[m-1])(X[m+n] - X[m+n-1])]$

$= r_{XX}[n] - r_{XX}[n-1] - r_{XX}[n+1] + r_{XX}[n]$

$= 2r_{XX}[n] - r_{XX}[n-1] - r_{XX}[n+1]$.

1. (continued)

$$r_{YY}[-2] = 2r_{XX}[-2] - r_{XX}[-3] - r_{XX}[-1]$$
$$= -0.5$$
$$r_{YY}[-1] = 2r_{XX}[-1] - r_{XX}[-2] - r_{XX}[0]$$
$$= 0$$
$$r_{YY}[0] = 2r_{XX}[0] - r_{XX}[-1] - r_{XX}[1]$$
$$= 1$$

By symmetry

$$r_{YY}[1] = r_{YY}[-1] = 0$$
$$r_{YY}[2] = r_{YY}[-2] = -0.5$$
$$\therefore r_{YY}[n] = \begin{cases} 1 & n=0 \\ 0 & |n|=1 \\ -0.5 & |n|=2 \\ 0 & \text{else} \end{cases}$$

b. $$\mu_Y[n] = E[Y[n]]$$
$$= E[X[2n]]$$
$$= 1$$
$$r_{YY}[n] = E[Y[m]Y[m+n]]$$
$$= E[X[2m]X[2m+2n]]$$
$$= r_{XX}[2n]$$
$$= \begin{cases} 1, & n=0 \\ 0, & \text{else} \end{cases}$$

c. $$\mu_Y[n] = E[Y[n]]$$
$$= \begin{cases} E[X[n/2]], & n/2 \text{ is an integer} \\ 0, & \text{else} \end{cases}$$
$$= \begin{cases} 1, & n/2 \text{ is an integer} \\ 0, & \text{else} \end{cases}$$
$$r_{YY}[m, m+n] = E[Y[m]Y[m+n]] = \begin{cases} E[X(\frac{m}{2})X(\frac{m+n}{2})] & m, n \text{ even} \\ 0 & \text{else} \end{cases}$$
$$= \begin{cases} r_{XX}(\frac{n}{2}) & m, n \text{ even} \\ 0 & \text{else} \end{cases}$$

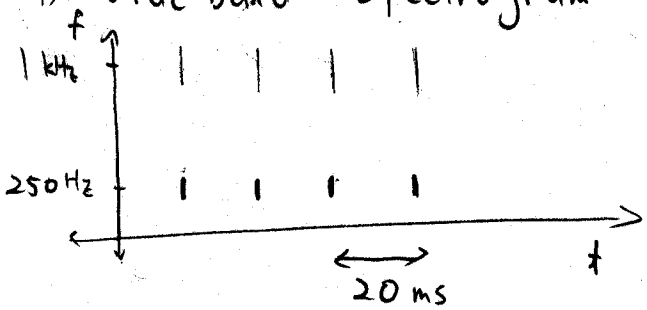
2. (20 pts) An individual with pitch frequency 50 Hz utters a voiced phoneme with a strong first formant at 250 Hz and a weaker second formant at 1 kHz.

   a. Sketch the wideband and narrowband spectrograms corresponding to this speech waveform. Be sure to label all important quantities.

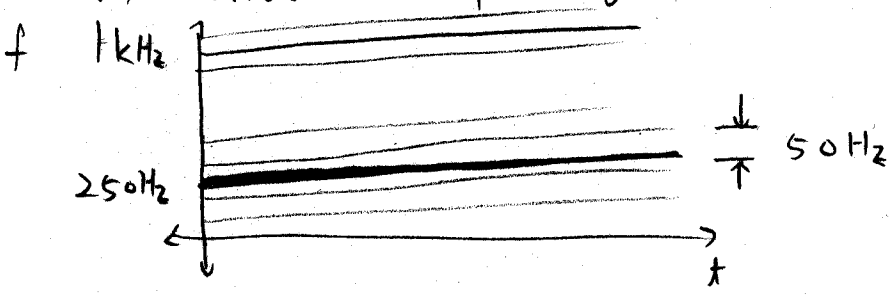
   b. Design a digital system consisting of a pulse generator driving a linear filter to synthesize this waveform. The system operates at an 8 kHz sampling rate. For this system, specify the pulse interval in samples and the approximate location in the Z plane of the poles for the filter.

a. Pitch period $P = 1/50 = 20$ ms

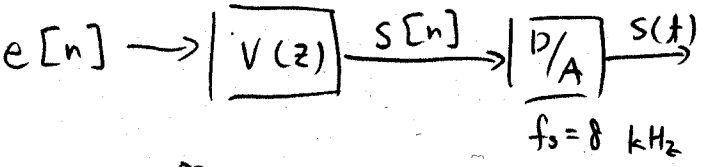
i) Wide band spectrogram



ii) Narrow band spectrogram



b. Pitch period $P = \frac{1}{50} \times 8000 = 160$ samples



$$e[n] = \sum_{k=-\infty}^{\infty} \delta[n - 160k]$$

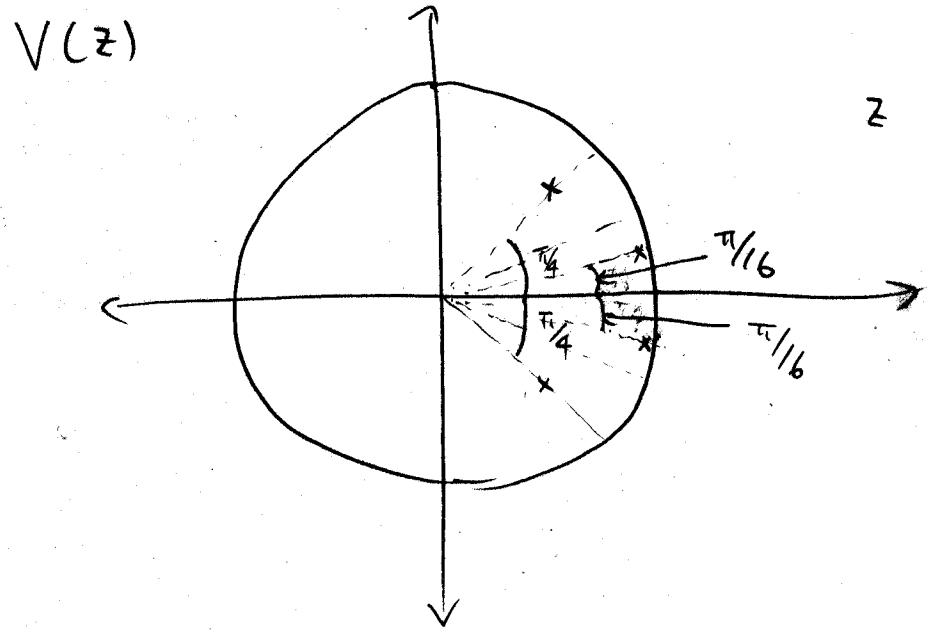
First formant frequency

$$\omega_1 = 2\pi \cdot \frac{250}{8000} = \pi/16$$

2. (continued)

Second formant frequency:

$$\omega_2 = 2\pi \cdot \frac{1000}{8000} = \pi/4$$



3 (25 pts.) In class, we defined the STDTFT of the signal $x[n]$ as

$$X(\omega, n) = \sum_k x[k]w[n-k]e^{-j\omega k}.$$

Suppose that we evaluate the STDTFT at L points $\omega_l = 2\pi l / L, l = 0,\ldots,L-1$ along the frequency axis; so

$$X[l,n] = \sum_k x[k]w[n-k]e^{-j2\pi lk/L}$$

a. Show that for each fixed value of l, $X[l,n]$ can be viewed as the output of a narrowband filter. Find an expression for the frequency response of this filter, and sketch what it would typically look like.

In class, we also showed that $x[n]$ could be reconstructed by summing the outputs of these L filters, provided the impulse responses $h_l[n]$ of the filters all had value $1/L$ at $n = 0$, and value 0 at $n = \pm L, \pm 2L, \pm 3L, \ldots$.

b. Show that a filter with frequency response

$$H(\omega) = \begin{cases} \frac{1}{L}(1 - |\omega / (2\pi / L)|), & |\omega| < 2\pi / L \\ 0, & \text{else} \end{cases}$$

satisfies this condition.

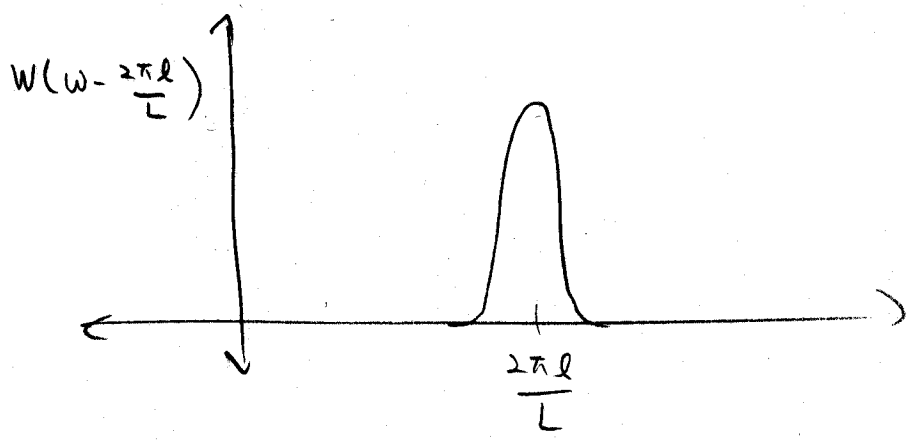
a. $X[l,n] = \sum_k x[k]\, w[n-k]\, e^{-j2\pi k/L}$

$= \sum_k x[n-k]\, w[k]\, e^{-j2\pi(n-k)/L}$

$= e^{-j2\pi ln/L}\left(\sum_k x[n-k]\, w[k]\, e^{j2\pi kl/L}\right)$

Narrow band filter

$w[k]\, e^{j2\pi lk/L} \xleftrightarrow{DTFT} W\left(\omega - \frac{2\pi l}{L}\right)$.



3. (continued)

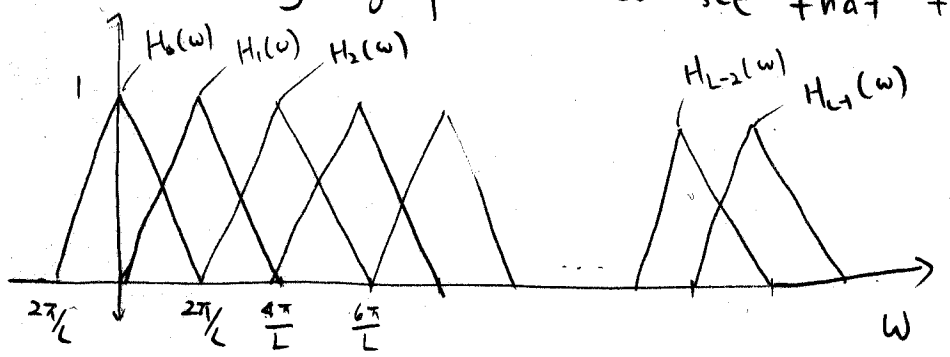
b. $H(\omega) = \begin{cases} 1 - |\omega/(2\pi/L)|, & |\omega| < 2\pi/L \\ 0, & \text{else} \end{cases}$

Solution I.

The condition for $h_\ell[n]$ for perfect reconstruction is equivalent to

$$\sum_{\ell=0}^{L-1} H\left(\omega - \frac{2\pi\ell}{L}\right) = 1$$

By the following graph we can see that the condition is satisfied.



$H_\ell(\omega) = H\left(\omega - \frac{2\pi\ell}{L}\right)$.

Solution II.

$$H(\omega) = \sqrt{L}\,\text{rect}\left(\frac{\omega}{2\pi/L}\right) * \sqrt{L}\,\text{rect}\left(\frac{\omega}{2\pi/L}\right)$$

$$\frac{1}{\sqrt{L}}\,\text{sinc}\left(\frac{\pi}{L}n\right) \xleftrightarrow{\text{DTFT}} \sqrt{L}\,\text{rect}\left(\frac{\omega}{2\pi/L}\right)$$

$$h_\ell[n] = \frac{1}{\sqrt{L}}\,\text{sinc}\left(\frac{\pi}{L}n\right) \times \frac{1}{\sqrt{L}}\,\text{sinc}\left(\frac{\pi}{L}n\right)$$

$$= \frac{1}{L}\,\text{sinc}^2\left(\frac{\pi}{L}n\right)$$

$h_\ell[0] = 1/L$

$h_\ell[kL] = 0$ where k is an integer and $k \neq 0$.

4. (25 pts.) Your grades on quizzes 1, 2, and 3 are 3, 4, and 7, respectively. Based on this data set, find the least squares prediction $\hat{g}[4]$ for your grade on quiz 4, where the predictor has the form $\hat{g}[n] = a_0 + a_1 n$. Here n denotes the index of the quiz; and the coefficients a_0 and a_1 are chosen to minimize

$$E = \sum_{n=1}^{3} \left[\hat{g}[n] - g[n]\right]^2,$$

where $g[n]$ is the grade for the n-th quiz.

$$E = \sum_{n=1}^{3} \left[\hat{g}[n] - a_0 - a_1 n\right]^2$$

$$\frac{\partial E}{\partial a_0} = 2 \sum_{n=1}^{3} \left(\hat{g}[n] - a_0 - a_1 n\right) = 0$$

$$3a_0 + a_1(1+2+3) = 3+4+7$$

$$3a_0 + 6a_1 = 14 \qquad -①$$

$$\frac{\partial E}{\partial a_1} = 2 \sum_{n=1}^{3} \left(\hat{g}[n] - a_0 - a_1 n\right) n = 0$$

$$a_0(1+2+3) + a_1(1^2+2^2+3^2) = 3\cdot 1 + 4\cdot 2 + 7\cdot 3$$

$$6a_0 + 14a_1 = 32 \qquad -②$$

From ① & ②

$$a_0 = 2/3, \quad a_1 = 2$$

$$\hat{g}[4] = a_0 + 4a_1$$

$$= 8 + 2/3$$

$$\approx 8.67$$