5.12 Find the decision regions that give the minimum average probability of error for the probabilities π_0 and π_1 if f_0 and f_1 are zero-mean Gaussian densities with *unequal* variances σ_0^2 and σ_1^2. Express Γ_0 and Γ_1 as unions of intervals.

HBP Problem 5.12

$$f_0(z) = \frac{1}{\sqrt{2\pi}\sigma_0} e^{-z^2/2\sigma_0^2} \qquad f_1(z) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-z^2/2\sigma_1^2}$$

π_0 $\qquad\qquad\qquad$ π_1

$$\Gamma_1 = \{z:\ \pi_1 f_1(z) \geq \pi_0 f_0(z)\}$$

$$\frac{\pi_1 e^{-z^2/2\sigma_1^2}}{\sigma_1} \geq \frac{\pi_0 e^{-z^2/2\sigma_0^2}}{\sigma_0} \iff \frac{\pi_1}{\pi_0}\frac{\sigma_0}{\sigma_1} \geq \exp\left\{-\frac{1}{2}\left(\frac{z^2}{\sigma_0^2} - \frac{z^2}{\sigma_1^2}\right)\right\}$$

$$\iff \ln\left(\frac{\pi_1}{\pi_0}\frac{\sigma_0}{\sigma_1}\right) \geq -\frac{1}{2}z^2\left(\frac{\sigma_1^2-\sigma_0^2}{\sigma_0^2\sigma_1^2}\right) \qquad (*)$$

$$\iff -2\ln\left(\frac{\pi_1}{\pi_0}\frac{\sigma_0}{\sigma_1}\right) \leq z^2\left(\frac{\sigma_1^2-\sigma_0^2}{\sigma_0^2\sigma_1^2}\right)$$

Case $\sigma_1 > \sigma_0$ $(*)$ becomes

$$z^2 \geq 2\frac{\sigma_0^2\sigma_1^2}{\sigma_1^2-\sigma_0^2}\ln\left(\frac{\pi_0}{\pi_1}\frac{\sigma_1}{\sigma_0}\right)$$

Subcase $\frac{\pi_0}{\pi_1}\frac{\sigma_1}{\sigma_0} \leq 1 \implies \ln\left(\frac{\pi_0}{\pi_1}\frac{\sigma_1}{\sigma_0}\right) \leq 0$

and condition $(*)$ defining Γ_1 is equiv. to $z^2 \geq 0$

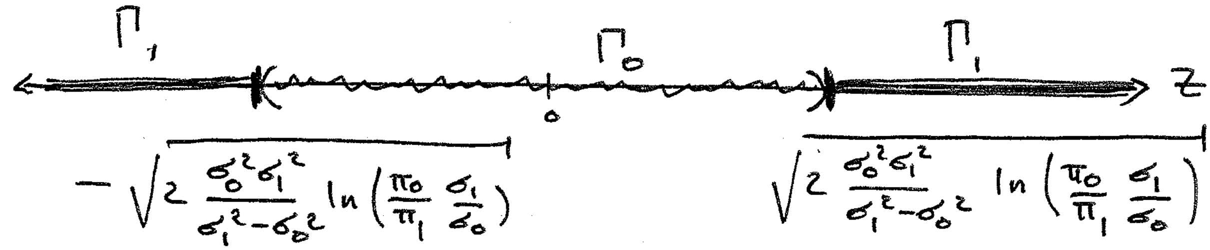
$\implies \Gamma_1 = (-\infty, \infty),\ \Gamma_0 = \phi$

Subcase $\frac{\pi_0}{\pi_1}\frac{\sigma_1}{\sigma_0} > 1$

$$(*) \iff z \geq \sqrt{2\frac{\sigma_0^2\sigma_1^2}{\sigma_1^2-\sigma_0^2}\ln\left(\frac{\pi_0}{\pi_1}\frac{\sigma_1}{\sigma_0}\right)}$$

or

$$z \leq -\sqrt{2\frac{\sigma_0^2\sigma_1^2}{\sigma_1^2-\sigma_0^2}\ln\left(\frac{\pi_0}{\pi_1}\frac{\sigma_1}{\sigma_0}\right)}$$



Case $\sigma_0 > \sigma_1$ (*) becomes

$$-2\ln\left(\frac{\pi_1}{\pi_0}\frac{\sigma_0}{\sigma_1}\right)\left(\frac{\sigma_0^2\sigma_1^2}{\sigma_1^2-\sigma_0^2}\right) \geq z^2$$

$\Longleftrightarrow$

$$z^2 \leq 2\frac{\sigma_0^2\sigma_1^2}{\sigma_0^2-\sigma_1^2}\ln\left(\frac{\pi_1}{\pi_0}\frac{\sigma_0}{\sigma_1}\right)$$

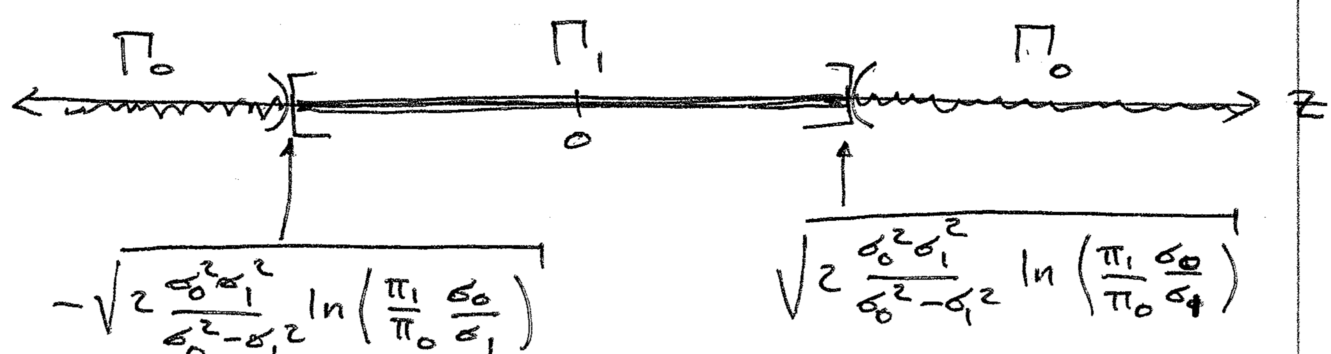
Subcase $\frac{\pi_1}{\pi_0}\frac{\sigma_0}{\sigma_1} \leq 1 \Longleftrightarrow \ln\left(\frac{\pi_1}{\pi_0}\frac{\sigma_0}{\sigma_1}\right) < 0$

and condition (*) defining Γ_1 is equiv. to $z^2 \leq$ a neg. number

$\Longrightarrow \Gamma_1 = \phi, \quad \Gamma_0 = (-\infty, \infty)$

Subcase $\frac{\pi_1}{\pi_0}\frac{\sigma_0}{\sigma_1} \geq 1 \Longleftrightarrow \ln\left(\frac{\pi_1}{\pi_0}\frac{\sigma_0}{\sigma_1}\right) \geq 0$

$$(*) \Longleftrightarrow -\sqrt{2\frac{\sigma_0^2\sigma_1^2}{\sigma_0^2-\sigma_1^2}\ln\left(\frac{\pi_1}{\pi_0}\frac{\sigma_0}{\sigma_1}\right)} \leq z \leq \sqrt{2\frac{\sigma_0^2\sigma_1^2}{\sigma_0^2-\sigma_1^2}\ln\left(\frac{\pi_1}{\pi_0}\frac{\sigma_0}{\sigma_1}\right)}$$



5.14 **(a)** Using Theorem 5-1 stated in Section 5.4, prove that the following randomized decision rule is a minimax decision rule for the densities f_0 and f_1 of Example 5-9. The decision rule is to observe z and

$$\begin{aligned} \text{if } f_1(z) &> 3f_0(z), \text{ decide 1 was sent;} \\ \text{if } f_1(z) &< 3f_0(z), \text{ decide 0 was sent;} \end{aligned}$$

and

$$\text{if } f_1(z) = 3f_0(z), \text{ make a random choice.}$$

The random choice is as follows: With probability $1/4$, decide 0 was sent, and with probability $3/4$, decide 1 was sent. In order to apply the theorem you must first show the decision rule is a Bayes decision rule. (What are the values of π_0 and π_1?) You must also prove that the decision rule has the property that $P_{e,0} = P_{e,1}$.

(b) Show that the following choice of Γ_0 and Γ_1 gives a nonrandomized minimax rule for the densities of Example 5-9:

$$\begin{aligned} \Gamma_0 &= [0, 5/4] \cup [2, 3]; \\ \Gamma_1 &= (5/4, 2). \end{aligned}$$

Is this decision rule a likelihood ratio test?

MBP Problem 5.14

Thm 5-1: If a decision rule is a Bayes rule and also satisfies $P_{e,0} = P_{e,1}$ then it is a minimax decision rule.

From Example 5-9: f_0 unif on $(0,3)$

f_1 unif on $(1,2)$

We are to consider the decision rule

$$f_1(z) \begin{array}{c} > \\ < \\ = \end{array} 3 f_0(z) \quad \begin{array}{l} \text{decide 1 was sent} \\ \text{decide 0 was sent} \\ \text{decide 0 with prob. } 1/4 \\ \text{and 1 with prob. } 3/4 \end{array}$$

(a)

Bayes rules are of the form

$$\Gamma_1 = \{ z \in \Gamma : \ \pi_1 f_1(z) > \pi_0 f_0(z) \}$$

$$\Gamma_0 = \{ z \in \Gamma : \ \pi_1 f_1(z) < \pi_0 f_0(z) \}$$

$$\Gamma_{\text{don't care}} = \{ z \in \Gamma : \ \pi_1 f_1(z) = \pi_0 f_0(z) \}$$

Thus must show a solution to

$$\frac{\pi_0}{\pi_1} = 3 \qquad \pi_0 + \pi_1 = 1 \qquad \pi_0, \pi_1 \geq 0$$

$$\Longrightarrow \pi_0 = 3\pi_1 \Longrightarrow 4\pi_1 = 1 \Longrightarrow \pi_1 = \frac{1}{4}, \ \pi_0 = \frac{3}{4}$$

So this is a Bayes rule for these priors.

Can we find the regions Γ_1, Γ_0 and $\Gamma_{\text{don't care}}$?

If instead define $\Gamma_0' = \Gamma_0 \cup \Gamma_{\text{don't care}}$

$$P_{e,0} = P(Z \in \Gamma_1 \mid H_0) = 0 \qquad (\Gamma_1 = \phi)$$

$$P_{e,1} = P(Z \in \Gamma_0' \mid H_1)$$
$$= P(0 < Z < 3 \mid H_1) = 1$$

$$\Longrightarrow \overline{P}_e^* = \frac{3}{4} \cdot 0 + \frac{1}{4} \cdot 1 = \frac{1}{4}$$

Note that we get same Bayes error in either of these two extreme cases, while the conditional error probs. behave very differently.

Let $\theta = 1/4$ be the prob. with which we decide H_0 when $Z \in \Gamma_{\text{don't care}}$.

Then

$$P_{e,0} = P(Z \in \Gamma_1 \mid H_0) + (1-\theta) P(Z \in \Gamma_{\text{don't care}} \mid H_0)$$
$$= P(Z \in \phi \mid H_0) + (1-\theta) P(1 \le Z \le 2 \mid H_0)$$
$$= 0 + (1-\theta)\frac{1}{3} = \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{4}$$

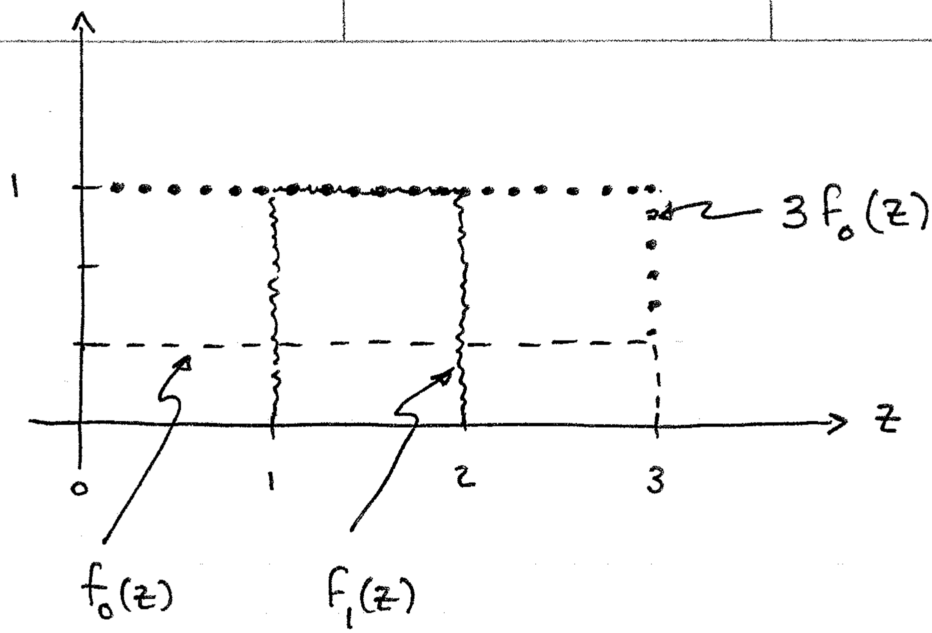
$$P_{e,1} = P(Z \in \Gamma_0 \mid H_1) + \theta P(Z \in \Gamma_{\text{don't care}} \mid H_1)$$

$$= P(0 < Z < 1 \mid H_1) + P(2 < Z < 3 \mid H_1)$$
$$+ \theta P(1 \le Z \le 2 \mid H_1)$$

$$= 0 + 0 + \theta \cdot 1 = \theta = \frac{1}{4}$$

$\therefore P_{e,0} = P_{e,1} = \frac{1}{4}$ hence minimax

Note that get the same $\overline{P}_e^*$ for this randomized rule.



Can restrict consideration to points z s.t $0 < z < 3$ since the interval encompasses the support of both pdfs.

Clearly

$$\Gamma_1 = \phi \quad \Gamma_0 = (0,1) \cup (2,3) \quad \Gamma_{\text{don't care}} = [1,2]$$

The usual Bayes error for the priors $\pi_0 = 3/4$, $\pi_1 = 1/4$ is

$$\overline{P_e^*} = \pi_0 P_{e,0} + \pi_1 P_{e,1}$$

but we do have to decide where we put $\Gamma_{\text{dont care}}$.

If define $\Gamma_1' = \Gamma_1 \cup \Gamma_{\text{dont care}}$

$$P_{e,0} = P(Z \in \Gamma_1' \mid H_0) = P(1 \le Z \le 2 \mid H_0) = 1/3$$

$$P_{e,1} = P(Z \in \Gamma_0 \mid H_1)$$

$$= P(0 < Z < 1 \mid H_1) + P(2 < Z < 3 \mid H_1) = 0$$

$$\Rightarrow \overline{P_e^*} = \frac{3}{4} \cdot \frac{1}{3} + \frac{1}{4} \cdot 0 = \frac{1}{4}$$

(b) Consider the nonrandomized decision rule

$$\Gamma_0 = [0, 5/4] \cup [2,3]$$
$$\Gamma_1 = (\tfrac{5}{4}, 2)$$

$$P_{e,0} = P(Z \in \Gamma_1 \mid H_0) = P(\tfrac{5}{4} < Z < 2 \mid H_0)$$
$$= \left(2 - \frac{5}{4}\right)\frac{1}{3} = \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{4}$$

$$P_{e,1} = P(Z \in \Gamma_0 \mid H_1) = P(0 \le Z \le \tfrac{5}{4} \mid H_1) + P(2 \le Z \le 3 \mid H_1)$$
$$= \left(\frac{5}{4} - 1\right) \cdot 1 + 0 = \frac{1}{4}$$

Clearly an equalizer rule. Also

$$\overline{P_e^*} = \pi_0 P_{e,0} + \pi_1 P_{e,1} = \frac{1}{4}$$

so also a Bayes rule. Hence is minimax.

This is not an LRT because we <u>cannot</u> find an η st

$$\Gamma_0 = \left\{ z : \frac{f_1(z)}{f_0(z)} < \eta \right\} \text{ and } \Gamma_1 = \left\{ z : \frac{f_1(z)}{f_0(z)} > \eta \right\}.$$

Note:

$$\frac{f_1(z)}{f_0(z)} = \begin{cases} 0 & 0 < z < 1 \\ 1/3 & 1 < z < 2 \\ 0 & 2 < z < 3 \end{cases}$$

5.15 A binary baseband data transmission system uses the antipodal signal set defined by

$$s_0(t) = \begin{cases} 2At/T, & 0 \le t < T/2, \\ A(2t - T)/T, & T/2 \le t < T, \\ 0, & \text{otherwise.} \end{cases}$$

The channel is an additive white Gaussian noise channel with noise spectral density $N_0/2$. The *minimax* criterion is to be used.

(a) What is the minimum probability of error for this system? Give your answer in terms of A, T, N_0, and the function Q.

(b) Give the impulse response of the filter that achieves the minimum error probability (i.e., the matched filter). Simplify your answer as much as possible.

(c) Give the optimum sampling time and optimum (minimax) threshold for the receiver that uses the filter of part **(b)**.

(d) Find the variance σ^2 of the output process when the input to the matched filter is a white-noise process with spectral density $N_0/2$.

(e) Suppose that the filter in the receiver is *not* the filter of part **(b)**, but is instead a filter with impulse response

$$h(t) = p_T(t).$$

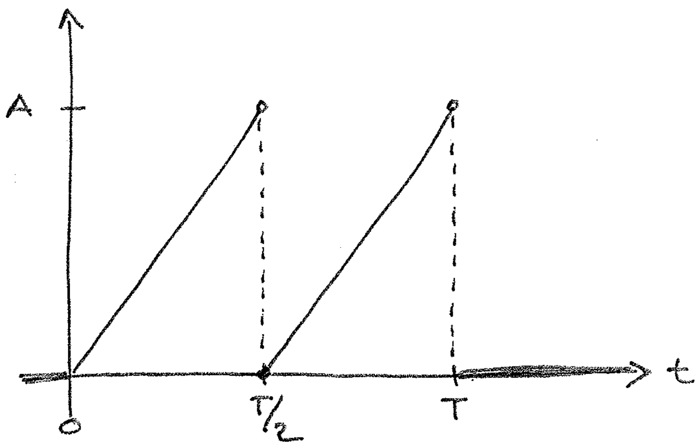
Give an expression for the output $\hat{s}_0(t)$ of this filter when the input is $s_0(t)$. Give the value of $\hat{s}_0(t)$ for *all* t in the range $-\infty < t < \infty$.

(f) For the filter of part **(e)**, find the maximum value of $\hat{s}(t)$, where the maximization is over all t in the range $-\infty < t < \infty$. (Note: This part can be solved independently of the solution to part **(e)**.)

MBP 5.15

Antipodal signalling $s_1(t) = -s_0(t)$ where

$$s_0(t) = \begin{cases} 2At/T & 0 \le t < T/2 \\ A(2t-T)/T & T/2 \le t < T \\ 0 & \text{else} \end{cases}$$



AWGN $N_0/2$

Use minimax criterion

For antipodal signals and zero mean additive Gaussion noise the minimax threshold is always

$$\gamma_m = 0$$

The minimum or best minimax error prob. occurs when the LTI filter is a matched filter

$$h(t) = c\left[s_0(T_0-t) - s_1(T_0-t)\right]$$

(b) $$= 2c\, s_0(T_0 - t)$$

any sampling time T_0 will work and the choice $T_0 = T$ is the smallest which results in a causal M.F.

From class or MBP formulas the max. SNR is

$$SNR_{max} = \frac{2\,\|s_0\|}{\sqrt{2N_0}}$$

$$\|s_0\|^2 = \int_0^T s_0^2(t)\,dt = 2\int_0^{T/2} s_0^2(t)\,dt = 2 \cdot \frac{4A^2}{T^2}\int_0^{T/2} t^2\,dt$$

$$\|s_o\|^2 = \frac{8A^2}{T^2}\,\frac{1}{3}\left(\frac{T}{2}\right)^3 = \frac{A^2 T}{3} \implies \|s_o\| = \frac{A\sqrt{T}}{\sqrt{3}}$$

$$SNR_{max} = \frac{2\cdot A\sqrt{T}}{\sqrt{3}}\,\frac{1}{\sqrt{2N_o}} = 2A\sqrt{\frac{T}{6N_o}}$$

$$P^*_{e,m} = Q\left(SNR_{max}\right) = Q\left(2A\sqrt{\frac{T}{6N_o}}\right).$$

(a) $$= Q\left(\sqrt{\frac{2A^2T}{3N_o}}\right) \longrightarrow = \min_h P^*_{e,m}$$

(c) As mentioned above $\gamma_m = 0$ and $T_o = T$.

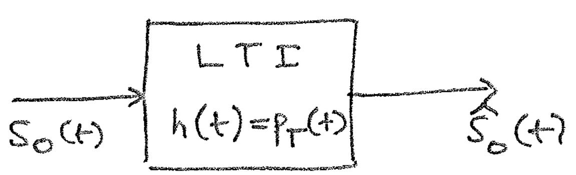
(d) Find variance of output process when input to filter is white Gaussian noise with $N_o/2$.

As shown in class and in MBP

$$\sigma^2 = \frac{N_o}{2}\|h\|^2 = \frac{N_o}{2}\,4c^2\|s_o\|^2$$

$$= 2N_o c^2 A^2 T/3$$

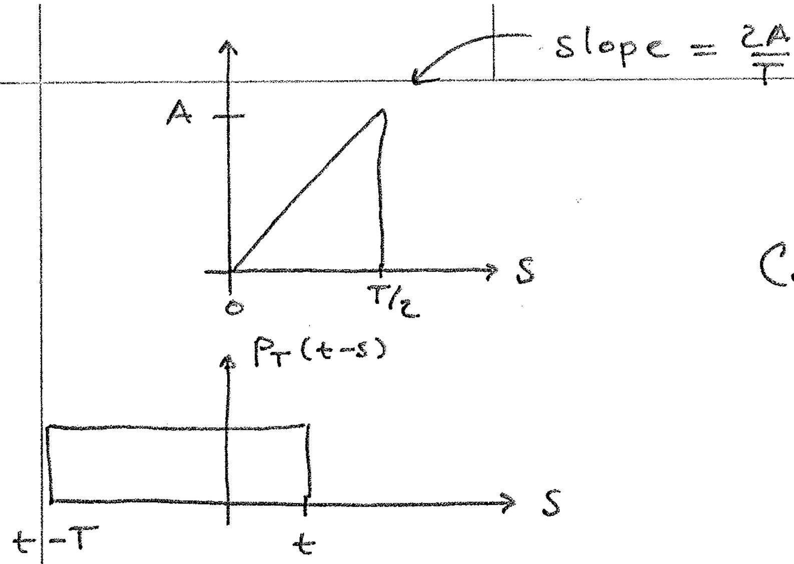
(e) Instead of using a MF let $h(t) = p_T(t)$.



So want to compute the convolution $\hat{s}_o(t) = p_T * s_o(t)$.

Using linearity and time invariance we first compute

Then can put pieces together to get final answer.



Call this partial result $\hat{\hat{S}}_o(t)$

From picture have cases.

Case: $t<0 \Longrightarrow \hat{\hat{S}}_o(t) = 0$

Case: $0<t<T/2 \Longrightarrow \hat{\hat{S}}_o(t) = \frac{1}{2} t \cdot \frac{2A}{T} t = \frac{A}{T} t^2$

Case: $T/2 < t < T \Longrightarrow \hat{\hat{S}}_o(t) = \frac{1}{2} \frac{T}{2} \cdot A = \frac{AT}{4}$

Case: $T < t < \frac{3T}{2} \Longrightarrow \hat{\hat{S}}_o(t) = \frac{AT}{4} - \frac{1}{2}(t-T) \frac{2A}{T} (t-T)$

$$= \frac{AT}{4} - \frac{A}{T}(t-T)^2$$

Case: $\frac{3T}{2} < t \Longrightarrow \hat{\hat{S}}_o(t) = 0.$

Then from linearity and time invariance

$$\hat{S}_o(t) = \hat{\hat{S}}_o(t) + \hat{\hat{S}}_o(t - T/2).$$

$$\hat{S}_o(t) = \begin{cases} 0 & t < 0 \\ \frac{A}{T} t^2 & 0 < t < T/2 \\ \frac{AT}{4} + \frac{A}{T}(t - T/2)^2 & \frac{T}{2} < t < T \\ \frac{AT}{2} - \frac{A}{T}(t-T)^2 & T < t < \frac{3T}{2} \\ \frac{AT}{4} - \frac{A}{T}\left(t - \frac{3T}{2}\right)^2 & \frac{3T}{2} < t < 2T \\ 0 & t > 2T \end{cases}$$

(f) Find max value of $\hat{S}_o(t)$.

From graphical picture of convolution and the positivity of s_o and p_T the max must occur at $t = T$

$$\max_t \hat{S}_o(t) = \frac{AT}{2}$$

5.18 Consider the communications system shown in Figure 5-4. The noise process $X(t)$ is a white Gaussian random process with spectral density $N_0/2$, and the signals $s_0(t)$ and $s_1(t)$ are given by

$$s_i(t) = (-1)^i A\, p_T(t)$$

for $i = 0$ and $i = 1$. The threshold is $\gamma = 0$ and the sampling time is $T_0 = \alpha T$ for $0 < \alpha < 2$. Investigate the effects of the sampling time by finding the error probabilities $P_{e,0}$ and $P_{e,1}$ in the following two cases:

(a) The filter is a linear time-invariant filter that is matched to the signals; that is, the impulse response is

$$h(\lambda) = s_0(T - \lambda) - s_1(T - \lambda).$$

Give $P_{e,0}$ and $P_{e,1}$ in terms of α, A, T, and N_0.

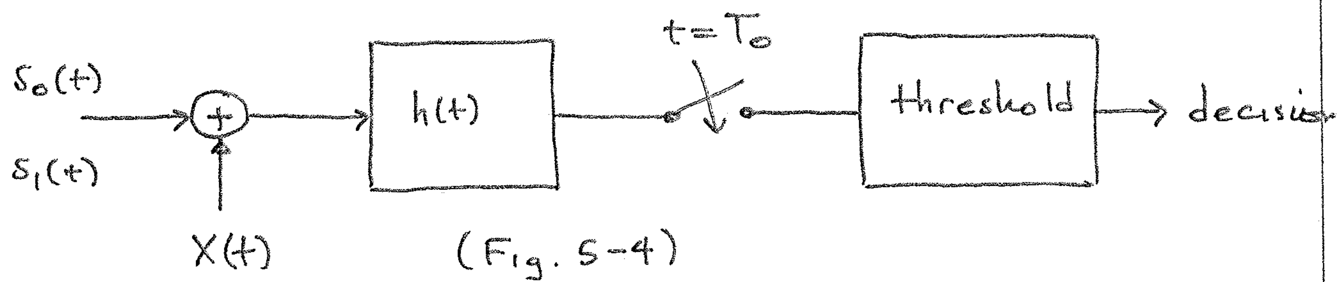
(b) The filter is an integrate-and-dump filter with output given by

$$Z(T_0) = \int_0^{T_0} Y(t)\, dt.$$

Give expressions for $P_{e,0}$ and $P_{e,1}$ in terms of α, A, T, and N_0.

(c) Express your answers to **(a)** and **(b)** in terms of α, $\mathcal{E}$, and N_0 (where $\mathcal{E}$ is the energy per pulse), and compare them.

MBP Problem 5.18



(Fig. 5-4)

white Gaussian $\frac{N_0}{2}$

$s_0(t) = A\,p_T(t) \qquad s_1(t) = -A\,p_T(t) \qquad$ antipodal

$\gamma = 0$

$T_0 = \alpha T \qquad 0 < \alpha < 2$

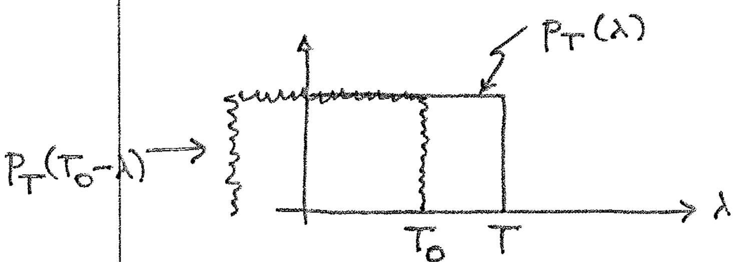
Find $P_{e,0}$ and $P_{e,1}$ assuming:

(a) $h(t) = s_0(T-t) - s_1(T-t)$

$$= 2A\,p_T(T-t) = 2A\,p_T(t)$$

Following approach taken in class

$$\mu_0 = s_0 * h(T_0) = \int s_0(\lambda)\, h(T_0-\lambda)\, d\lambda = 2A^2 \int p_T(\lambda)\, p_T(T_0-\lambda)\, d\lambda$$



$$\mu_0 = 2A^2 \int_0^{T_0} d\lambda \qquad 0 < T_0 < T$$

$$= 2A^2 T_0 = 2A^2 \alpha T \qquad 0 < \alpha \le 1$$

$$\mu_0 = 2A^2 \int_{T_0 - T}^{T} d\lambda = 2A^2 (2T - T_0) \qquad T < T_0 < 2T$$

$$= 2A^2 (2-\alpha) T \qquad 1 \le \alpha < 2$$

$$\therefore\ -\mu_1 = \mu_0 = \begin{cases} 2A^2 \alpha T & 0 < \alpha \le 1 \\ 2A^2 (2-\alpha) T & 1 \le \alpha < 2 \end{cases}$$

Note that the minimax threshold $= 0$, which agrees with choice made in this problem.

For AWGN

$$\sigma^2 = \frac{N_0}{2}\int h^2(t)\,dt = \frac{N_0}{2}4A^2\int_0^T dt = 2A^2N_0T$$

$$\therefore \text{SNR} = \frac{\mu_0-\mu_1}{2\sigma} = \frac{4A^2\alpha T}{2\sqrt{2A^2N_0T}} \qquad 0<\alpha\le 1$$

$$= \frac{4A^2(2-\alpha)T}{2\sqrt{2A^2N_0T}} \qquad 1\le\alpha<2$$

Simplifying

$$\text{SNR} = \begin{cases} A\sqrt{\frac{2T}{N_0}}\,\alpha & 0<\alpha\le 1 \\ A\sqrt{\frac{2T}{N_0}}\,(2-\alpha) & 1\le\alpha<2 \end{cases}$$

Since minimax $P_{e,0} = P_{e,1} = Q(\text{SNR})$.

(b) Integrate and dump with

$$Z(T_0) = \int_0^{T_0} Y(t)\,dt = \int_0^{T_0} s_i(t)\,dt + \int_0^{T_0} X(t)\,dt$$

For the signal part consider $i=0$

$$\int_0^{T_0} s_0(t)\,dt = A\int_0^{T_0} p_T(t)\,dt = \begin{cases} AT_0 & 0<T_0\le T \\ AT & T<T_0<2T \end{cases}$$

$$= \begin{cases} A\alpha T & 0<\alpha\le 1 \\ AT & 1\le\alpha<2. \end{cases}$$

Also $\int_0^{T_0} s_1(t)\,dt = -\int_0^{T_0} s_0(t)\,dt.$

For the noise part note that $\int_0^{T_0} X(t)\,dt$ is a Gaussian r.v. with zero mean and variance

$$\sigma^2 = E\left\{\left(\int_0^{T_0} X(t)\,dt\right)^2\right\} = \int_0^{T_0}\int_0^{T_0} E\{X(t)X(\lambda)\}\,dt\,d\lambda$$

$$= \frac{N_0}{2}\int_0^{T_0}\int_0^{T_0}\delta(t-\lambda)\,dt\,d\lambda = \frac{N_0}{2}\int_0^{T_0} d\lambda = \frac{N_0 T_0}{2}$$

$$= \frac{N_0}{2} T\alpha$$

So decision problem for this statistic can be written

$$Z(T_0) \sim \begin{cases} N\left(\mu(\alpha), \frac{N_0 T}{2}\alpha\right) & \text{under } H_0 \\ N\left(-\mu(\alpha), \frac{N_0 T}{2}\alpha\right) & \text{"} \quad H_1 \end{cases}$$

where

$$\mu(\alpha) = \begin{cases} AT\alpha & 0<\alpha\le 1 \\ AT & 1<\alpha<2 \end{cases}$$

$\gamma = 0$ is minimax threshold as before. The SNR depends on α

$$SNR = \frac{\mu_0 - \mu_1}{2\sigma} = \frac{\mu_0}{\sigma}$$

$$= \begin{cases} \frac{AT\alpha}{\sqrt{N_0 T\alpha/2}} & 0<\alpha\le 1 \\ \frac{AT}{\sqrt{N_0 T\alpha/2}} & 1<\alpha<2 \end{cases} = \begin{cases} A\sqrt{\frac{2T}{N_0}\alpha} & 0<\alpha\le 1 \\ A\sqrt{\frac{2T}{N_0\alpha}} & 1<\alpha<2 \end{cases}$$

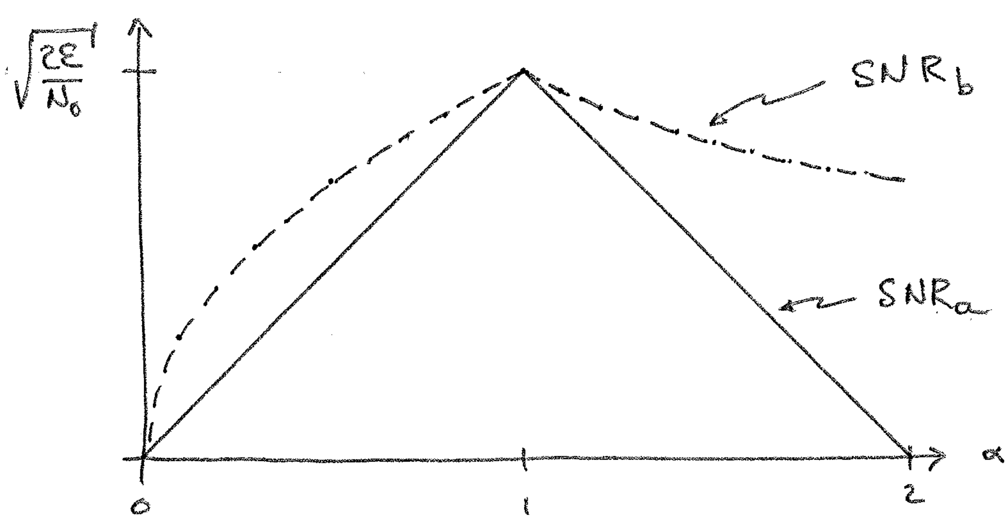
$P_{e,0} = P_{e,1} = Q(SNR)$.

(c) Express prev. answers in terms of $\mathcal{E}$ (energy per pulse).

$$\mathcal{E} = \int_0^T A^2\,dt = A^2 T$$

$$SNR_a = \begin{cases} \sqrt{\frac{2\mathcal{E}}{N_0}}\,\alpha & 0 < \alpha \le 1 \\ \sqrt{\frac{2\mathcal{E}}{N_0}}\,(2-\alpha) & 1 < \alpha < 2 \end{cases}$$

$$SNR_b = \begin{cases} \sqrt{\frac{2\mathcal{E}}{N_0}}\,\sqrt{\alpha} & 0 < \alpha \le 1 \\ \sqrt{\frac{2\mathcal{E}}{N_0}}\,\frac{1}{\sqrt{\alpha}} & 1 < \alpha < 2 \end{cases}$$



5.21 The signal set $\{s_0, s_1\}$ is antipodal and $s_0(t) = \exp(-t^2)$ for all t. The channel is an AGN channel with zero-mean, wide-sense stationary noise having spectral density $S_X(\omega)$. The received signal plus noise is filtered with a time-invariant linear filter. The output of the filter is sampled at time T_0 and compared with a threshold $\gamma = 0$ in the decision device. Assume that the noise is white with spectral density $S_X(\omega) = N_0/2$.

- **(a)** What is the impulse response of the optimum filter?
- **(b)** If the optimum filter is employed, what is the probability of error for this system?
- **(c)** For the filter you obtained in part **(a)**, what is the signal component at the output of the filter at the sampling time if $s_0(t)$ was actually transmitted?

5.22 Repeat Problem 5.21**(a)** under the assumptions of that problem, except that the noise now has spectral density

$$S_X(\omega) = 1/(\omega^2 + \alpha^2), \quad -\infty < \omega < \infty,$$

where α is real and positive.

MBP 5.21 (not assigned)

$$s_0(t) = e^{-t^2} \quad t \in \mathbb{R} \qquad \text{(Signals not time limited)}$$
$$= -s_1(t)$$

$$S_X(\omega) = \frac{N_0}{2} \quad \text{(AWGN)}$$

Filter sampled at $t = T_0$. Threshold $\gamma = 0$.

(a) Opt. filter is a MF however cannot be made causal since signals s_0 & s_1 are not time limited.

$$h_{MF}(t) = c\left[s_0(T_0-t) - s_1(T_0-t)\right] = 2c\, s_0(T_0-t)$$
$$= 2c\, e^{-(T_0-t)^2}$$

(b) + (c) Compute minimax error prob. and signal component @ MF output at time T_0 given s_0.

SNR for MF is (antipodal, AWGN)

$$SNR = \sqrt{\frac{2\bar{\mathcal{E}}}{N_0}}$$

$$\bar{\mathcal{E}} = \mathcal{E}_0 = \mathcal{E}_1 = \int_{-\infty}^{\infty} \left(e^{-t^2}\right)^2 dt = \underbrace{2\int_0^{\infty} e^{-2t^2}\, dt}$$

Need to express as a Gaussian integral for evaluation.

Recall

$$\frac{1}{\sqrt{2\pi}\,\sigma}\int_0^{\infty} e^{-x^2/2\sigma^2}\, dx = \frac{1}{2}$$

$$\int_0^{\infty} e^{-2t^2} dt = \int_0^{\infty} e^{-t^2/0.5}\, dt = \frac{\sqrt{2\pi}\,(0.5)}{\sqrt{2\pi}\,(0.5)}\int_0^{\infty} e^{-t^2/2(0.5)^2} dt$$
$$= \sqrt{2\pi}\,(0.5)\,\frac{1}{2}$$

Therefore

$$\overline{\mathcal{E}} = 2\left[\sqrt{2\pi}\,(0.5)\,\frac{1}{2}\right] = \frac{\sqrt{2\pi}}{2} = \sqrt{\frac{\pi}{2}}$$

$$\Longrightarrow \frac{2\overline{\mathcal{E}}}{N_0} = \frac{2}{N_0}\sqrt{\frac{\pi}{2}} = \frac{\sqrt{2\pi}}{N_0} \Longrightarrow SNR = \sqrt{\frac{2\overline{\mathcal{E}}}{N_0}} = \sqrt{\frac{\sqrt{2\pi}}{N_0}}$$

$$P^*_{e_{gm}} = Q(SNR) = Q\left(\sqrt{\frac{\sqrt{2\pi}}{N_0}}\right).$$

MF output @ $t = T_0$

$$s_0 * h(T_0) = \int s_0(\lambda)\,h(T_0-\lambda)\,d\lambda = \int s_0(\lambda)\,2c\,s_0(\lambda)\,d\lambda$$

$$= 2c\int s_0^2(\lambda)\,d\lambda = 2c\,\overline{\mathcal{E}}$$

$$= 2c\sqrt{\frac{\pi}{2}} = \sqrt{2\pi}\,c$$

MBP 5.22

Same as before except let $S_x(\omega) = \frac{1}{\omega^2 + \alpha^2}$ where $\alpha > 0$.

For colored noise case we found the MF to be

$$H_{MF}(\omega) = c e^{-j\omega T_0} \frac{S_0^*(\omega) - S_1^*(\omega)}{S_x(\omega)}$$

$$= \frac{2c e^{-j\omega T_0} S_0^*(\omega)}{S_x(\omega)} = \frac{\tilde{H}_{MF}(\omega)}{S_x(\omega)}$$

where $\tilde{H}_{MF}(\omega)$ is the MF from problem 5.21.

$$H_{MF}(\omega) = (\omega^2 + \alpha^2)\tilde{H}_{MF}(\omega)$$

$$\updownarrow$$

$$h_{MF}(t) = -\frac{d^2}{dt^2}\tilde{h}_{MF}(t) + \alpha^2 \tilde{h}_{MF}(t)$$

Since $-(j\omega)^2 = \omega^2$ and differentiation in the time domain corresponds to mult. by $j\omega$ in the freq. domain.

$$\tilde{h}_{MF}(t) = 2c\, e^{-(T_0-t)^2}$$

$$\tilde{h}'_{MF}(t) = 2c\, e^{-(T_0-t)^2} \cdot 2(T_0-t) = 4c\,(T_0-t)\, e^{-(T_0-t)^2}$$

$$\tilde{h}''_{MF}(t) = -4c\, e^{-(T_0-t)^2} + 4c\,(T_0-t)\, e^{-(T_0-t)^2} \cdot 2(T_0-t)$$

$$= -4c\, e^{-(T_0-t)^2} + 8c\,(T_0-t)^2\, e^{-(T_0-t)^2}$$

$$= 4c\, e^{-(T_0-t)^2}\left[2(T_0-t)^2 - 1\right]$$

$\therefore$

$$h_{MF}(t) = -\tilde{h}''_{MF}(t) + \alpha^2 \tilde{h}_{MF}(t)$$

$$= 4c\, e^{-(T_0-t)^2}\left[1 - 2(T_0-t)^2\right] + 2c\,\alpha^2 e^{-(T_0-t)^2}$$

(a) $$= 2c\, e^{-(T_0-t)^2}\left[2\left(1-2(T_0-t)^2\right) + \alpha^2\right]$$