c) Find the correlation coefficient of U and V.

$$\rho_{UV} = \frac{Cov[U,V]}{\sqrt{Var[U]\,Var[V]}}$$

$$Cov[U,V] = Cov[2X+Y,\ 3X-2Y]$$
$$= Cov[2X,3X] + Cov[2X,-2Y]$$
$$+ Cov[Y,3X] + Cov[Y,-2Y]$$
$$= 6\,Var[X] - 4\,\cancel{Cov[X,Y]}^{\,0}$$
$$+ 3\,\cancel{Cov[Y,X]}^{\,0} - 2\,Var[Y]$$
$$= 6 \cdot 9 - 2 \cdot 16 = 22$$

$$\rho_{UV} = \frac{22}{\sqrt{52 \cdot 145}} = 0.2533$$

Functions of Two R.V.s

Often, we are interested in one or more functions of r.v.s from a random experiment.

One Function of Two R.Vs

Let X, Y be r.v.s ~~with~~ and let $Z = g(X, Y)$.

Would like to find $f_Z(z)$ (or $p_Z(z_K)$) in terms of $g(x, y)$ and $f_{X,Y}(x, y)$ (or $P_{X,Y}(x_i, y_j)$).

Distribution Method

We can use the distribution method as before.

$$
\begin{aligned}
F_Z(z) &= \Pr(Z \leq z) \\
&= \Pr(g(X,Y) \leq z) \\
&= \Pr((X,Y) \in \{(x,y) : g(x,y) \leq z\}) \\
&= \iint\limits_{g(x,y) \leq z} f_{X,Y}(x,y)\, dx\, dy
\end{aligned}
$$

Then find

$$
f_Z(z) = \frac{d}{dz} F_Z(z)
$$

<u>Note:</u> $\{(x, y) : g(x, y) \leq z\}$ now defines a <u>region</u> in the xy-plane

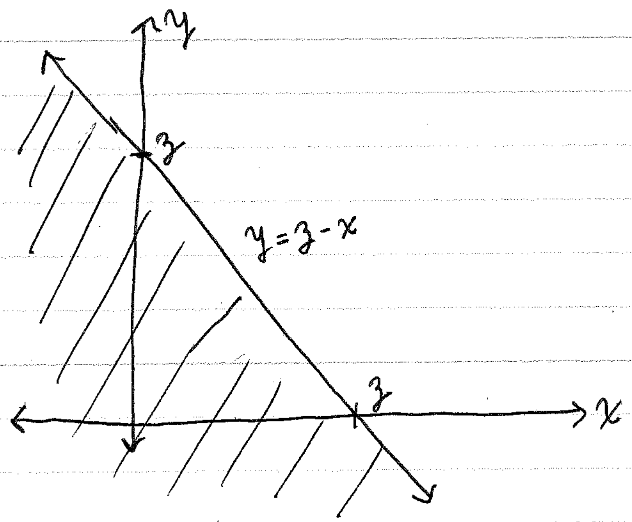
$\underline{\text{Ex}}$ Let $Z = X + Y$. Find cdf and pdf of Z.

<u>Distribution Method</u>

$$F_Z(z) = \Pr(Z \le z)$$

$$= \Pr(X + Y \le z)$$

$$= \Pr(Y \le z - X)$$



$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_{X,Y}(x,y)\, dy\, dx$$

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

$$= \frac{d}{dz} \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_{X,Y}(x,y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} \frac{d}{dz} \int_{-\infty}^{z-x} f_{X,Y}(x,y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} f_{X,Y}(x, z-x) \overbrace{\frac{d}{dz}(z-x)}^{1} \, dx$$

$$= \int_{-\infty}^{\infty} f_{X,Y}(x, z-x) \, dx$$

<u>Note</u>: If X, Y are independent and $Z = X+Y$, then

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x) \, dx$$

$$= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) \, dx$$

$$= f_X(z) * f_Y(z)$$

<u>EX</u>. Let X, Y be independent exponential r.v.s with mean 1. Let $Z = X+Y$. Find $f_Z(z)$.

$$f_X(x) = e^{-x} u(x) \quad , \quad f_Y(y) = e^{-y} u(y)$$

$$f_Z(z) = f_X(z) * f_Y(z)$$

$$= \int_{-\infty}^{\infty} e^{-x} u(x) \, e^{-(z-x)} u(z-x) \, dx$$

$$= \int_0^z e^{-z} \, dx \quad , \quad z \geq 0$$

$$= z e^{-z} \quad , \quad z \geq 0$$

$$= 0 \quad , \quad \text{else}$$

<u>Ex</u> (HW 4, Problem 2)

A binary transmission system transmits a signal X that is either -1 or 1, with $\Pr(X=-1) = 2/3$. The received signal is $Y = X + N$, where N is a Gaussian r.v. with mean 0 and variance σ^2. Assume X, N are independent. Find $f_Y(y)$.

$$f_Y(y) = f_X(y) * f_N(y)$$

$$f_N(n) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-n^2}{2\sigma^2}\right)$$

$$f_X(x) = \sum_{x_i} \Pr(X = x_i)\,\delta(x - x_i)$$

$$= \Pr(X=-1)\,\delta(x+1) + \Pr(X=1)\,\delta(x-1)$$

$$= 2/3\,\delta(x+1) + 1/3\,\delta(x-1)$$

$$f_Y(y) = f_N(y) * \left(2/3\,\delta(y+1) + 1/3\,\delta(y-1)\right)$$

$$= 2/3\, f_N(y+1) + 1/3\, f_N(y-1)$$

$$= 2/3 \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(y+1)^2}{2\sigma^2}\right)$$

$$+ 1/3 \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-1)^2}{2\sigma^2}\right)$$

If X, Y are discrete r.v.s and $Z = g(X,Y)$, then we can find $P_Z(z)$ as

$$P_Z(z_k) = \Pr(Z = z_k)$$
$$= \Pr(g(X,Y) = z_k)$$
$$= \sum\sum_{g(x_i, y_j) = z_k} P_{X,Y}(x_i, y_j)$$

Two Functions of Two R.V.s

Let X, Y be r.v.s and let

$$U = g(X,Y), \quad V = h(X,Y)$$

Want to find $f_{U,V}(u,v)$ (or $P_{U,V}(u_k, v_l)$) in terms of $g(x,y)$ and $h(x,y)$ and $f_{X,Y}(x,y)$ (or $P_{X,Y}(x_i, y_j)$)

Case 1: X, Y discrete r.v.s then

$$P_{U,V}(u_k, v_l) = \Pr(U = u_k, V = v_l)$$
$$= \Pr(g(X,Y) = u_k, h(X,Y) = v_l)$$
$$= \sum\sum_{\substack{g(x_i, y_j) = u_k \\ h(x_i, y_j) = v_l}} P_{X,Y}(x_i, y_j)$$