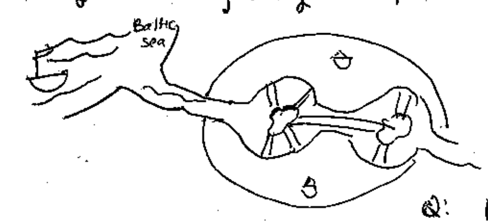
20th March 2012

GRAPH THEORY

History: Königsberg (present Russia, then Germany)

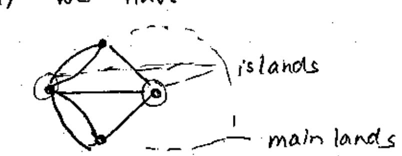


two islands in the river with bridges as shown

Q: Is it possible to go on a walk in Königsberg using every bridge exactly once and to return to the starting point?

Supposedly, Euler was prompted by this question to invent the graph theory.

Reducing the problem, we have



a graph is a collection of vertices and edges (= a pair of vertices)

2 flavors: (1) Basic Type = Un directed
= DEFAULT

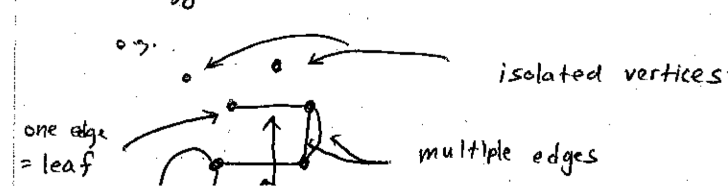
(2) Directed

= edge has a start and the end and we can only use the edge in the prescribed direction.

Graphs appear in:

→ Scheduling
→ road networks (weighted to compensate for length)
→ airline connections
→ any networks (social, electrical)
→ polyhedral e.g. 2-D rep of cube (3-D)

Terminology:



with six vertices, we say

set of vertices $|V| = 6$

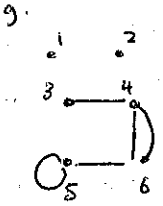
Definitions → A Simple graph is a graph without loops or multiple edges

→ Two vertices are adjacent if joined by an edge

→ Adjacency Matrix $(|V| \times |V|)$ has entries

i^{th}, j^{th} entry = # of edges between the i^{th} + j^{th} vertex

e.g.



adjacency matrix

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 0 \end{pmatrix}$$

✩ Symmetric

if the graph is directed, need not be symmetric..

eg. directed graph

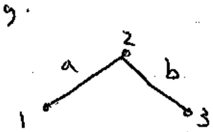
$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

the convention is: +1 for "in"
-1 for "out"

→ incidence matrix records

in $(|E| \times |V|)$ matrix...

e.g.

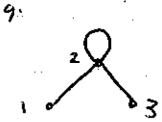


$$\begin{array}{c} a \\ b \end{array} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

records which vertex belongs to which edge

→ degree of vertex = # of edges involving the vertex

e.g.

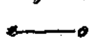


deg(1) = 1 = deg(3)

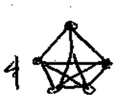
deg(2) = ~~2~~ 4 (loop counts twice)

If all vertices of G have the same ~~number of vertices~~ degrees, say k,
then G is called k-regular

eg.

1-regular 

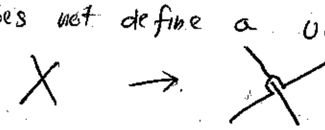
3-regular 

3-regular , 4 

Note: ~~crossection~~

cross-section does not define a vertices...

think



Q: Given a degree sequence, can one produce a graph that realizes it?

e.g.

a. $\{1, 1, 2, 2, 3\}$? $\rightarrow$??

b. $\{1, 2, 2, 2, 3\}$? $\rightarrow$ yes:

<u>Thm</u> Handshake Lemma

= the sum of the degrees in graph is even

Proof: Imagine each edge as a handshake. Each handshake constitutes two handshake, one by each ~~pair~~ vertices in a pair.

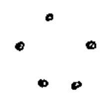
$$\sum(\text{all handshakes}) = 2 \cdot |E| = \sum \text{all degree}$$

If the degree sum is even, the following gives a G w/ the right sequence.

(1) order the degree by size

$n_1, n_2, \dots, n_k \qquad n_i \leq n_{i+1}$

(2) Draw blank graph w/ K vertices



K=5

(3) Pick either $a \geq 2$ or "$a \geq 1$" and make a corresponding edge = reiterate

e.g. $\{1, 2, 2, 2, 3\}$

(1) Blank graph

(2) Pick 1 and 2 = $\{0, 1, 2, 2, 3\}$

(3) —— " —— = $\{0, 0, 1, 2, 3\}$

(4) —— " —— = $\{0, 0, 0, 1, 3\}$

(5) —— " —— = $\{0, 0, 0, 0, 2\}$

(6) Finish w/ loop

<u>Handshake Lemma for Directed Graphs</u>

$$\sum \text{in-degree} = |E| = \sum(\text{out-degree})$$

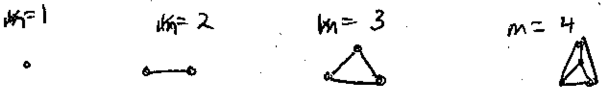
$\deg^+(v) \qquad \deg^-(v)$

<u>Some types of graphs</u>

- Complete graphs "K_m"

K = "Komplete" (german) m = number of vertices

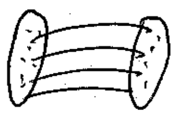
e.g. $m=1 \quad m=2 \quad m=3 \quad m=4$



least # of connectivity to connect all vertices

• bipartite

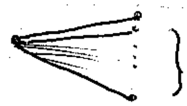
"think of transatlantic flight, from US + Europe, but no domestic, intra-continent



• bipartite complete

$K_{m,n}$ (m,n corresponds to number of elements in each "continent")

eg $K_{1,n}$



} n vertices

$K_{2,2}$

• tree graph

- simple (no loops or multiple edges)
- no interesting circuits

⇒ circuits are sequences of edges such, that the first vertex of the next edge = 2^{nd} vertex of the current edge.

ex

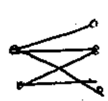
interesting circuits

uninteresting ... a wasted step.

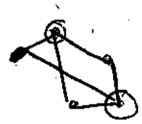
e.g of trees

Q: How do you test whether a given graph is bipartite?

→ bipartite graphs may not be obvious



← obvious



← not so obvious

to check ... we could naively check for all possibilites

$$\sum_{k} \binom{|V|}{k}$$ = ~~the~~ number of cases to check for bipartite graph..

pick $0=k$, $1=k$, $2=k$...

see if condition is satisfied

$= 2^{|V|}$ (consider each vertices is in a same group or not...)

not only is this impractical ... <u>undoable</u>

Better Approach:

Pick a vertex. and color it "blue". $\wedge$

All vertices connected to the "blue" vertex must be "red"

" "red" vertex must be "blue"

Reiterate

2 Possible Ends:

→ a contradiction (vertex = red and blue) = G is not bipartite

→ all vertices are colored w/o contradiction = G is bipartite.