e.g. For $t \in [-\frac{1}{2}, \frac{1}{2}]$, $s(t) = rect(2t) - \frac{1}{2}$, $T_0 = 1$

$$s(t) = \sum_{k=-\infty}^{\infty} S_k e^{j2\pi kt}$$

when $k \neq 0$ $$S_k = \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} s(t) e^{-j\frac{2\pi kt}{T_0}} dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} [rect(2t) - \frac{1}{2}] e^{-j2\pi kt} dt$$

$$= -\frac{1}{2}\int_{-\frac{1}{2}}^{-\frac{1}{4}} e^{-j2\pi kt} dt + \frac{1}{2}\int_{-\frac{1}{4}}^{\frac{1}{4}} e^{-j2\pi kt} dt - \frac{1}{2}\int_{\frac{1}{4}}^{\frac{1}{2}} e^{-j2\pi kt} dt$$

$$= \frac{1}{j4\pi k}(e^{j\frac{\pi k}{2}} - e^{j\pi k}) + \frac{-1}{j4\pi k}(e^{-j\frac{\pi k}{2}} - e^{j\frac{\pi k}{2}}) + \frac{1}{j4\pi k}(e^{-j\pi k} - e^{-j\frac{\pi k}{2}})$$

$$= \frac{1}{j2\pi k}(e^{j\frac{\pi k}{2}} - e^{-j\frac{\pi k}{2}}) + \frac{1}{j4\pi k}(e^{-j\pi k} - e^{j\pi k}) = \begin{cases} \frac{1}{\pi k}\sin\frac{\pi k}{2} & , k \text{ is odd} \\ 0 & , k \text{ is even} \end{cases}$$

when $k = 0$. $$S_0 = \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} s(t)\, dt = 0$$

$$s(t) = \frac{1}{T_0} S_0 + \frac{1}{T_0} \sum_{\substack{k=-\infty \\ k \text{ is odd}}}^{\infty} S_k e^{j2\pi kt}$$

$$= \sum_{\substack{k=-\infty \\ k \text{ is odd}}}^{\infty} \frac{1}{\pi k} \sin\frac{\pi k}{2} \cdot \cos 2\pi kt + \sum_{\substack{k=-\infty \\ k \text{ is odd}}}^{\infty} \frac{1}{\pi k} \sin\frac{\pi k}{2} \cdot j\sin 2\pi kt$$

$$= \sum_{\substack{k=1 \\ k \text{ is odd}}}^{\infty} \frac{2}{\pi k} \sin\frac{\pi k}{2} \cos 2\pi kt$$

$$= \sum_{\substack{k=1 \\ k \text{ is odd}}}^{\infty} \frac{2\sin\frac{\pi k}{2}}{\pi k} \sin(2\pi kt + \frac{\pi}{2})$$

Fourier Series Expansion

$x(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T}$ , where $X_k$ is the fourier coefficients given by

$$X_k = + \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-j2\pi kt/T} dt$$

Special Case $X_0 = \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,dt$

Usually, the positive half and negative half can be combined together to $\cancel{X_k'}$

$$\cancel{x(t) = \frac{1}{T}X_0 + \sum_{k=1}^{\infty} X_k' e^{j2\pi kt/T} = \frac{1}{T}X_0 + \sum_{k=1}^{\infty} X_k'[\cos(2\pi kt/T) + j\sin(2\pi kt/T)]}$$

Further simplified the expansion into form

$$x(t) = a_0 + \sum_{k=1}^{\infty} A_k \sin(2\pi kt/T + \theta_k), \quad a_0 = \frac{1}{T}X_0, \quad f = \frac{1}{T}$$

e.g. $S(t) = rect(t - \frac{1}{2})$, $T_0 = 2$, ~~$t \in [0,2]$~~ For $t \in [0,2]$

$$S(t) = \frac{1}{2}\sum_{k=-\infty}^{\infty} S_k e^{j\pi kt}$$

when $k \neq 0$ $\quad S_k = \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} s(t) e^{-j\frac{2\pi kt}{T_0}} dt = \int_{-1}^{1} rect(t - \frac{1}{2}) e^{-j\pi kt} dt = \int_0^1 e^{-j\pi kt} dt$

$$= \frac{-1}{j\pi k}(e^{-j\pi k} - 1) \quad (k \neq 0)$$

when $k = 0$

$$S_0 = \int_0^1 1\,dt = 1$$

$$= \begin{cases} \frac{2}{j\pi k} & k \in odd \\ 0 & k \in even \end{cases}$$

$$S(t) = \frac{1}{2}S_0 + \frac{1}{2}\sum_{\substack{k=-\infty \\ k \in odd}}^{\infty} \frac{2}{j\pi k} e^{j\pi kt}$$

$$= \frac{1}{2} + \frac{1}{2}\sum_{\substack{k=-\infty \\ k \in odd}}^{\infty} \frac{2}{j\pi k}\cos\pi kt + \frac{1}{2}\sum_{\substack{k=-\infty \\ k \in odd}}^{\infty} \frac{2}{j\pi k} j\sin\pi kt$$

$$= \frac{1}{2} + \frac{1}{2}\sum_{\substack{k=1 \\ k \in odd}}^{\infty} 2 \times \frac{2}{j\pi k} \cdot j\sin\pi kt$$

$$= \frac{1}{2} + \sum_{\substack{k=1 \\ k \in odd}}^{\infty} \frac{2\sin\pi kt}{\pi k}$$