(Remind) CT FS.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t) \quad , \quad \phi_k(t) = e^{j 2\pi \frac{k}{T} t}$$

$$, \quad a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-j 2\pi \frac{k}{T} t} \, dt$$

* energy of a signal : $E_x = \int_{-\infty}^{\infty} x^2(t) \, dt$

More generally, (for complexed-value signal)

$$x(t) = \underset{\text{In-phase}}{x_I(t)} + j \underset{\text{quadrature-phase}}{x_Q(t)}$$

$$= \underset{\text{real part}}{x_r(t)} + j \underset{\text{imaginary part}}{x_i(t)}$$

$Re\{x(t)\} = \frac{1}{2}\{x(t) + x^*(t)\}$ $\quad$ $Im\{x(t)\} = \frac{1}{2}\{x(t) - x^*(t)\}$

$$\rightarrow \text{Energy} = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \underbrace{\int_{-\infty}^{\infty} x_r^2(t) \, dt}_{\text{energy of real part}} + \underbrace{\int_{-\infty}^{\infty} x_i^2(t) \, dt}_{\text{energy of imaginary part}}$$

- Energy for sinewaves is typically computed as energy per period:

$$|s_k(t)|^2 = s_k(t)\, s_k^*(t)$$

$$E_k = \frac{1}{T}\int_{-T/2}^{T/2} |s_k(t)|^2\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} e^{j2\pi\frac{k}{T}t}\, e^{-j2\pi\frac{k}{T}t}\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} e^{0}\, dt = \frac{1}{T}\cdot T = 1$$

- With a complex-amplitude: energy of $a_k s_k(t)$ is

$|a_k|^2$: energy of $a_k e^{j2\pi\frac{k}{T}t}$

: energy in k-th harmonic term in FS expansion with freq. $\frac{k}{T}$ (Hz)

- Energy per period of periodic signal
 = sum of energy per period in each of the harmonic sinewaves in the FS expansion of $x(t)$

$$\frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

(proof) using orthogonality of the sinewaves,

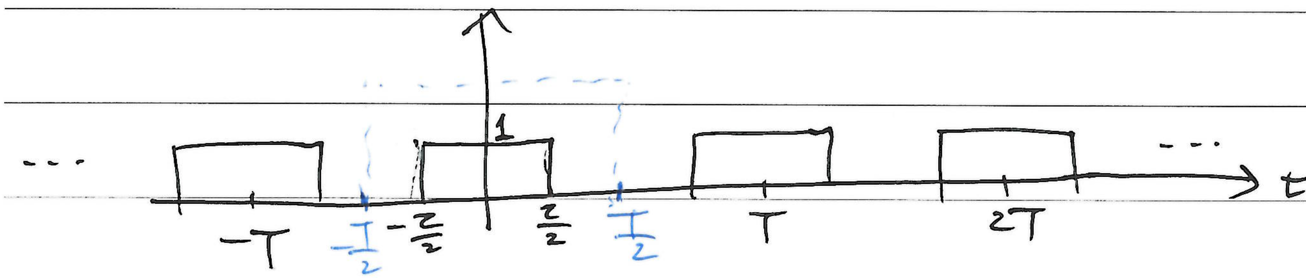
$$\frac{1}{T}\int_{-T/2}^{T/2} x(t)\, x^*(t)\, dt = \frac{1}{T}\int_{-T/2}^{T/2} \sum_k a_k s_k(t) \sum_l a_l^* s_l^*(t)\, dt$$

$$= \sum_k \sum_l a_k a_l^* \underbrace{\frac{1}{T}\int_{-T/2}^{T/2} s_k(t) s_l^*(t)\, dt}_{\frac{1}{T}\cdot T\,\delta[k-l]} = \sum_{k=-\infty}^{\infty} |a_k|^2$$

◦ Some basic Fourier series

1. Periodic train of rectangular pulses

$$x(t) = \sum_{n=-\infty}^{\infty} \mathrm{rect}\left(\frac{t-nT}{\tau}\right) = \sum_{k=-\infty}^{\infty} a_k s_k(t)$$



$$a_k = \text{"}\frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j2\pi\frac{k}{T}t}\, dt\text{"} = \frac{1}{T}\int_{-\tau/2}^{\tau/2} e^{-j2\pi\frac{k}{T}t}\, dt$$

$$= \frac{1}{\cancel{T}} \cdot \frac{1}{-j2\pi\frac{k}{\cancel{T}}} e^{-j2\pi\frac{k}{T}t}\Big|_{-\tau/2}^{\tau/2} = \frac{1}{k\pi}\cdot\frac{-1}{2j}\underbrace{\left\{ e^{-j\frac{\cancel{2}\pi k\tau}{\cancel{2}T}} - e^{+j\frac{\cancel{2}\pi k\tau}{\cancel{2}T}} \right\}}_{= -2j\sin\left(\frac{k\pi\tau}{T}\right)}$$

(Euler's rule)

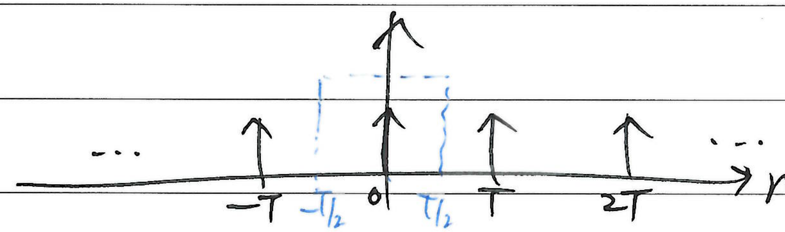
$$= \frac{\sin\left(k\pi\frac{\tau}{T}\right)}{k\pi}, \quad -\infty < k < \infty$$

at $k=0 \Rightarrow \frac{\tau}{T}$

2. Periodic Train of Dirac Delta Func.

⇒ very Important for our theoretical analysis of sampling theory later in Chap. 7.

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$



$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j2\pi\frac{k}{T}t}\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \underline{\delta(t)}\, e^{-j2\pi\frac{k}{T}t}\, dt = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{0}\, dt$$

(has value only when $t=0$)

$$= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, dt = \frac{1}{T}$$

$$\Rightarrow x(t) = \boxed{\sum_{n=-\infty}^{\infty} \delta(t-nT) = \sum_{k=-\infty}^{\infty} \underbrace{\frac{1}{T}}_{a_k}\, \underbrace{e^{+j2\pi\frac{k}{T}t}}_{s_k(t)}}$$

Some key Properties of FS

"$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi \frac{k}{T} t}$"

- Time-shift property

$$y(t) = x(t-t_0) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi \frac{k}{T}(t-t_0)}$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{\left(a_k e^{-j2\pi k \frac{t_0}{T}}\right)}_{\text{new FS coeff. for } y(t)} e^{j2\pi \frac{k}{T} t}$$

- Time-scaling

$$y(t) = x(at) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi \frac{k}{T} at}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{\frac{j2\pi kt}{T/a}} \Rightarrow \text{new period for } y(t) : \frac{T}{a}$$

$a_k \downarrow$ (FS coeff is the same)

- Differentiation

$$y(t) = \frac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} \underbrace{\left(a_k j 2\pi \frac{k}{T}\right)}_{\text{new FS coeff for } y(t)} e^{j2\pi \frac{k}{T} t}$$

○ Basic Property : $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi \frac{k}{T} t}$

$y(t) = \sum_k b_k e^{j2\pi \frac{k}{T} t}$] same period.

$$z(t) = A\,x(t) + B\,y(t)$$

$$= \sum_k \underbrace{(A a_k + B b_k)}_{\text{new FS coeff.}} \underline{e^{j2\pi \frac{k}{T} t}}$$

○ For additional properties of FS, see Table 3.1 (pg 206)

$* \quad \sum_k = \sum_{k=-\infty}^{\infty}$

* Basic DT FS property

(Recall) $x[n] = \sum_{k=0}^{N-1} \underline{a_k}\, \underline{e^{j 2\pi \frac{k}{N} n}}$

◦ time-shift

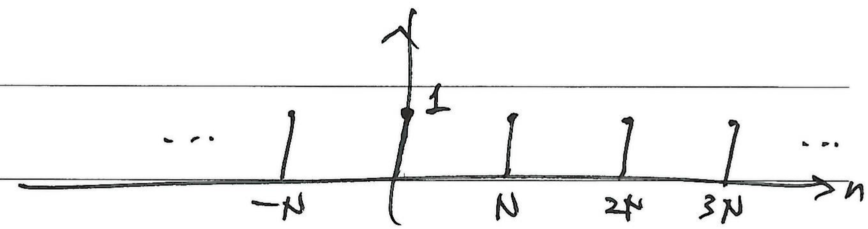
$y[n] = x[n-n_0] = \sum_{k=0}^{N-1} a_k e^{j 2\pi \frac{k}{N}(n-n_0)}$

$= \sum_{k=0}^{N-1} \underbrace{\left(a_k e^{-j 2\pi \frac{k}{N} n_0}\right)}_{\text{new FS coeff.}} \underline{e^{j 2\pi \frac{k}{N} n}}$

◦ other properties of DT FS are listed in Table 3.7 (pg. 221)

- Basic FS for DT train of Kronecker Delta Func.

$$x[n] = \sum_{l=-\infty}^{\infty} \delta[n - lN]$$



FS coeff. : $a_k = \frac{1}{N}$, $k = 0, 1, \cdots, N-1$

$$x[n] = \boxed{\sum_{l} \delta[n - lN] = \sum_{k=0}^{N-1} \frac{1}{N} e^{j 2\pi \frac{k}{N} n}}$$

(proof)

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j 2\pi \frac{k}{N} n} = \frac{1}{N} \sum_{n=0}^{N-1} \delta[n] e^{-j 2\pi \frac{k}{N} n}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \delta[n] e^{j0} = \frac{1}{N}$$

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