HOMEWORK 4 **DUE: SEPTEMBER 18, 2007**

REQUIRED PROBELMS. General: 1, 2, 8, 9, 12, 18, 22
Pole Placement: 1, 5

Before beginning, carefully read through your notes and be able to reproduce all derivations without your notes.

TRUE-FALSE

1. A valid 2x2 state transformation is $\begin{bmatrix} 1 & t \\ 1 & 1-t \end{bmatrix} z(t) = x(t)$. _____

GENERAL PROBLEMS

1. (Review. Complete response) Find the complete state-response and the complete output-response of the state model below when x(0) = $[1 \quad 1]^T$ and $u(t) = t1^+(t)$. What happens as t --> ∞?

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [1 \quad 0] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

2. (State Transformations*) (a) Let x be the state vector of the 3-dimensional observable canonical form. Let z be the state vector of the 3-dimensional controllable canonical form. Determine the entries of the nonsingular state transformation, Tz = x, relating the two canonical forms.
(b) Repeat for the 2-dimensional canonical form.
(c) Describe the set of all state transformation matrices T for which A_{ccf} (the A-matrix of the 3x3 controllable canonical form) is similar to A_{ocf} (the A-matrix of the 3x3 observable canonical form).

3. (State model equivalence, controllable form, pole placement) A state model for a particular system is

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [1 \quad -1] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [1]u(t)$$

(a) Determine the input-output differential equation of the system. Hint: form a set of matrix equations of the form

$$\begin{bmatrix} y(t) \\ \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} u(t) \\ \dot{u}(t) \\ \ddot{u}(t) \end{bmatrix}$$

Using row operations produce a zero row at the bottom of the matrix multiplying the x-vector. What is the result?

(b) Carefully and fully derive the controllable canonical form of the differential equation

$$D^2 y + a_1 Dy + a_2 y = b_2 u + b_1 Du + b_0 D^2 u$$

and put in matrix form. Let z be the vector of state variables.

(c) Your answer to part (a) should have the form given in part (b) for appropriate coefficients a_i and b_i. Find the controllable canonical form of the original system state model? Choose z as the new state variable designation.

(d) Find state feedback which will assign the poles, $\lambda_1 = -1$ and $\lambda_2 = -2$, to the system when it is in the controllable canonical state model form.

(e) Find a state transformation, Tz = x, which relates the given system state variables, x, as given above to the controllable canonical state variables, z.

(f) Show that the observable canonical form of the given state model is

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + [1]u$$

4. (Controllable form, pole placement*) Let D denote differentiation. **(a)** Show that the controllable canonical form of the differential equation

$$D^2 \hat{y} + a_1 D\hat{y} + a_2 \hat{y} = u$$

is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\hat{y} = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]u$$

(b) Determine a new output equation for the state model of (a) in terms of a new variable y when u and y satisfy

$$D^2 y + a_1 Dy + a_2 y = b_2 u + b_1 Du + b_0 D^2 u$$

(c) If $a_1 = a_2 = 3$, determine state feedback u = Fx so that the new poles/eigenvalues of the system are at $\lambda_1 = -1$ and $\lambda_2 = -1$.

5. (Observable form, pole placement*) Repeat problem 4 for the observable form, i.e., construct the observable form of the differential equation given in (2-b). Then assuming that $a_1 = a_2 = 3$ and $b_2 = 2$, $b_1 = 1$, and $b_0 = -1$, determine state feedback u = Fx so that the new poles/eigenvalues of the system are at $\lambda_1 = -1$ and $\lambda_2 = -1$.

6. (Controllable form, pole placement*) Let D be the derivative operator.
(a) Construct the controllable canonical form of the differential equation

$$D^3y + a_1D^2y + a_2Dy + a_3y = b_3u + b_2Du + b_1D^2u + b_0D^3u$$

(b) What are the natural frequencies of the system when $a_1 = 2$, $a_2 = 2$, and $a_3 = 1$.
(c) Find state feedback to assign new poles at $-2 \pm j, -1$.

7. (Observable form, transfer function) Let D be the derivative operator. Consider the differential equation model of a system:

$$D^3y + a_1D^2y + a_2Dy + a_3y = b_3u + b_2Du + b_1D^2u + b_0D^3u$$

(a) Derive the observable canonical form of the differential equation. Hint: you must have a non-zero D-matrix in the resulting state model.
(b) Find the natural frequencies of the system when $a_1 = 0$, $a_2 = -1$, and $a_3 = 0$ and compute the system transfer function.

8. (Controllable form*) Consider the 4-th order differential equation below where D represents the derivative operator, i.e. $D^n y$ is the n-th derivative of y.

$$D^4\hat{y} + a_1D^3\hat{y} + a_2D^2\hat{y} + a_3D\hat{y} + a_4\hat{y} = u$$

(a) By inspection what is the controllable canonical form for this state model?
(b) Derive this form from the differential equation.
(c) For the given differential equation show that if $D^n u$ is the input then the response is $D^n \hat{y}$ assuming no initial conditions and appropriate differentiability.
(d) (Rehash of linearity) Now show that if the response to u_1 is $\hat{y}_1$ and the response to u_2 is $\hat{y}_2$, then the response to $(b_1u_1 + b_2u_2)$ is $(b_1\hat{y}_1 + b_2\hat{y}_2)$.
(e) Determine the controllable canonical form of the differential equation

$$D^4y + a_1D^3y + a_2D^2y + a_3Dy + a_4y = b_4u + b_3Du + b_2D^2u + b_1D^3u + b_0D^4u$$

9. (Observable form*) Find the observable canonical form of the differential equaiton

$$D^4y + a_1D^3y + a_2D^2y + a_3Dy + a_4y = b_4u + b_3Du + b_2D^2u + b_1D^3u + b_0D^4u$$

10. (Observable form, pole placement) (a) Find the observable canonical form of the scalar differential equation

$$D^4y + a_1D^3y + y = -u + D^4u$$

(b) If $a_1 = 0$, determine state feedback which will assign the characteristic polynomial $\lambda^4 + \lambda^2 + 1$ to the system as follows:

(i) What is the form of the state feedback matrix F.

(ii) Derive the effect of the state feedback matrix on the system A-matrix.

(iii) Compute the characteristic polynomial of the system with state feedback F included. Be careful. Expand along the easyest columns.

(iv) Find the appropriate state feedback F to assign the desired characteristic polynomial.

11. (Controllable canonical form, time varying system) Investigate constructing the controllable canonical form of the differential equation

$$D^3y(t) + a_1(t)D^2y(t) + a_2(t)Dy(t) + a_3(t)y(t) = u(t)$$

Here, the bottom row of A(t) should have time varying entries and the B-matrix should be the same as in the time-invariant case. How would you modify, if possible, the construction when the right-hand side is (i) $b_3(t)\,u(t)$ and (ii) $b_3(t)\,u(t) + b_2(t)\,Du(t)$?

12. (Transfer functions and state space realization) Construct **(a)** the controllable, and **(b)** the observable canonical state space realizations of the transfer function

$$H(s) = \frac{y(s)}{u(s)} = \frac{b_0s^3 + b_1s^2 + b_2s + b_3}{s^3 + a_1s^2 + a_2s + a_3}$$

(c) (Challenge) If

$$H(s) = \frac{y(s)}{u(s)} = \frac{4}{(s+1)(s+2)(s+3)}$$

find a state space realization in which the A-matrix is A = diag(-1, -2, -3). Is this diagnonal realization unique? Explain.

(d) (Challenge continued) Find a diagonal realization of the transfer function

$$H(s) = \frac{y(s)}{u(s)} = \frac{4(s^2+1)}{(s+1)(s+2)(s+3)}$$

13. (Transfer functions and state space realization) A transfer function of a system is of the form $H(s) = H_1(s)\, H_2(s)$ where

$$H_i(s) = \frac{b_0^i s^2 + b_1^i s + b_2^i}{s^2 + a_1^i s + a_2^i}$$

Here H(s) is said to be a product or cascade of second order sections.

(a) Construct state space realizations (in the controllable canonical form) of $H_1(s)$ and $H_2(s)$.
(b) Construct a state space realization of H(s) that reflects the cascade structure of H(s).

14. (Conversion to Controllable Canonical Form) Consider the state dynamics $\dot{x} = Ax + Bu$ where A is nxn and B is nx1. Let $Q = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$ and suppose $\det[Q] \neq 0$. The following algorithm will convert the given state dynamics to the controllable canonical form[1].

(a) Compute Q^{-1}. Designate the last row by the row-vector v^T.
(b) Form the matrix

$$V = \begin{bmatrix} v^T \\ v^T A \\ \vdots \\ v^T A^{n-1} \end{bmatrix}$$

and compute V^{-1}.

(c) The (algebraically) equivalent system $\dot{z} = VAV^{-1}z + VBu$ is in the controllable canonical form.
(d) Show that if $\hat{F}$ is a state feedback matrix that assigns poles in the z-coordinates, then $\hat{F}$ V assigns this same set of poles in the original x-coordinates.
(e) Prove that the state transformation defined in steps (a) and (b), does, in fact transform the system to the controllable canonical form. In showing that the new A-matrix is in the controllable canonical form, you will need to use the Caley-Hamilton Theorem to demonstrate the proper structure of the last row of the new A-matrix. The **Caley-Hamilton Theorem** says that if $\pi_A(\lambda)$ is the characteristic polynomial of A, then $\pi_A(A) = [0]$, the zero-matrix.

15. (State Model Construction) A differential equation model for a magnetic microphone is

[1] For several algorithms see J. D. Aplevich, "Direct Computation of Canonical Forms for Linear Systems by Elementary Matrix Operations," IEEE Trans. Auto. Control, April 1974, pp. 124-126.

$$L\frac{di}{dt} + (R_c + R)i + B\frac{dz}{dt} = 0$$

$$M\frac{d^2z}{dt^2} + \mu\frac{dz}{dt} + Kz - Bi = F$$

where I is the current through the coil and Z is the displacement of the diaphragm element. The input is the force F on the diaphragm, and the output is the voltage drop RI. Suppose that $R_c = 9.0\ \Omega$, $R = 1.0\ \Omega$, $L = 10^{-3}$ H, M = 0.01 kg, m = 0.1 N-sec/m, K = 0.5 N/m, and B = 0.3 V-sec/m.
(a) Find a state model in the controllable canonical form for the magnetic microphone.
(b) Using the duality between the observable and controllable forms, find a state model in the observable canonical form.

16. (Linearization and controllable/observable form) The preditor-prey model described in chapter 1 is:

$$\frac{dH}{dt} = a_1 H - c_1 PH$$

$$\frac{dP}{dt} = -a_2 P + c_2 PH$$

where (i) a_1, a_2, c_1, c_2 are all positive values, (ii) a_1H is the uninhibited host's growth rate, (iii) $-c_1PH$ represents the decrease in growth rate due to the presence of parasites, (iv) $-a_2P$ represents the natural death rate of the parasites, and (v) the product term, c_2HP, models the dependence of the growth rate of the parasites upon the existence of a host.
(a) Determine the nonzero constant equilibrium solutions for H and P.
(b) Linearize the system about these nonzero constant equilibrium solutions. Specify the linearized perturbation state model of the system.
(c) Compute the controllable canonical form of the linearized model.
(d) Compute the observable canonical form of the linearized model.

17. (Linearization and controllable/observable forms*) Define $a = g/L > 0$, $b = k/m \geq 0$, and $c = 1/(mL^2) > 0$, then one can write the equations of motion of a pendulum as

$$\ddot{\theta} = -a\ \sin(\theta) - b\ \dot{\theta} + c\ u(t)$$

where θ is the angle subtended by the rod and the vertical axis and L is the length of the rod of the pendulum. The input u(t) = T(t) is a torque applied to the rod at the point of rotation.
(a) Choose state variables and write a nonlinear state model for the pendulum equation.
(b) Linearized the model about the angle $\theta = \theta^*$ where θ^* is a fixed angle away from the vertical. Find the associated constant input, $u^*(t) = T^*$, necessary to maintain the pendulum at the desired angle θ^*. Then determine the linearized equations about these nominal constant solutions.
(c) Find the controllable and observable forms of the state model of (b).

18. (Controllable form and response calculation) Given the controllable canonical form of a third order state dynamics, i.e.,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

it is possible to factor the A-matrix as $A = T\,D\,T^{-1}$ whenever the poles of the system (eigenvalues of A) are distinct. If λ_1, λ_2, and λ_3 are the distinct eigenvalues of the A-matrix, then

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix}^{-1}$$

Here, because of its structure, T is called a **Vandermonde matrix** of order 3.

(a) Show, in general, that $\exp(At) = T \exp(Dt)\, T^{-1}$.

(b) For the state dynamics

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

compute the factorization, $A = TDT^{-1}$.

(c) If $x(0) = [\lambda_1, \lambda_2, \lambda_3]^T$, compute the zero-input response.

(d) If $u(t) = 1^+(t)$, compute the zero-state response.

(e) Compute the complete response when $x(0) = [\lambda_1, \lambda_2, \lambda_3]^T$ and $u(t) = 1^+(t)$.

19. (Observable form and response calculation) Given the observable canonical form of the third order discrete-time state dynamics

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_3 \\ b_2 \\ b_1 \end{bmatrix} u$$

then whenever the eigenvalues, λ_1, λ_2, and λ_3, of the A-matrix are distinct

$$A=\begin{bmatrix}0 & 0 & -a_3\\ 1 & 0 & -a_2\\ 0 & 1 & -a_1\end{bmatrix}=T^{-1}DT=\begin{bmatrix}1 & \lambda_1 & \lambda_1^2\\ 1 & \lambda_2 & \lambda_2^2\\ 1 & \lambda_3 & \lambda_3^2\end{bmatrix}^{-1}\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}\begin{bmatrix}1 & \lambda_1 & \lambda_1^2\\ 1 & \lambda_2 & \lambda_2^2\\ 1 & \lambda_3 & \lambda_3^2\end{bmatrix}$$

(a) Compute the above factorization for A when

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & -500\\ 0 & 1 & -60\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 12.5\\ 0.25\end{bmatrix}u$$

(b) If $x(0) = [0, -500, -60]^T$, compute the zero-input response. Simplify.

(c) If $u(t) = e^{-10t}\, 1^+(t)$, compute the zero-state state-response.

(d) Compute the complete state-response when $x(0) = [0, -500, -60]^T$ and $u(t) = e^{-10t}\, 1^+(t)$.

20. True-False: Consider the linear differential equation

$$D^2y+3Dy+2y=-2u-Du+D^2u$$

1. The D-matrix in a controllable canonical form realization is D = [1]. _____
2. The C-matrix in a controllable canonical form realization is C = [-2 -1]. _____
3. A state transformation relating the controllable and observable canonical forms of the above is nonsingular. _____

21. (Time varying state transformation and zero-input response computation) Consider the time varying homogeneous state dynamics

$$\dot{x}=\begin{bmatrix}t & 1-t^2\\ 1 & -t\end{bmatrix}x$$

(a) Show that the time varying state transformation

$$T(t)z(t)=\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}z(t)=x(t)$$

leads to a time invariant set of dynamics $\dot{z}=Mz$ for an appropriate constant matrix M. Specify M.

(b) Use T(t) and $\dot{z}=Mz$ with appropriate z(0) to compute x(t) given $x(0) = [1 \quad 1]^T$.

22. Consider the time varying homogeneous state dynamics

$$\dot{x} = \begin{bmatrix} t-2 & 1 \\ -1-t^2 & -t-2 \end{bmatrix} x + \begin{bmatrix} 1 \\ -t \end{bmatrix} u(t)$$

(a) Show that the time varying state transformation

$$T(t)z(t) = \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix} z(t) = x(t)$$

leads to a time invariant set of dynamics $\dot{z} = \hat{A}z + \hat{B}u$.

(b) From the results of part (a), find z(1) and compute z(t) given that x(1) = $[1 \quad 1]^T$. Now determine the zero input response $x_{zi}(t)$.

(c) Determine the zero-state sate response for z(t) and then x(t) when u(t) = t $1^+(t)$.

23. (Read ahead problem) Let D be the derivative operator. (a) Determine the observable canonical form of the differential equation

$$D^4 y - y = u - b_3 Du - 0.5D^2 u + 2D^3 u + D^4 u$$

(b) If $D^3y(1) = D^2y(1) = D^1y(1) = y(1) = 1$, and u(t) = 10 t exp(-t) $1^+(t)$, determine x(1) so that the response for t ≥ 1 of the state model and the differential equation will coincide.

(c) If possible, determine an impulsive input which will set up the initial condition, $x(1^+)$, computed in (b) assuming $x(1^-) = 0$ and $b_3 = 1$. (This is Chap 5 material.)

24. Consider the time varying homogeneous state dynamics

$$\dot{x} = \begin{bmatrix} t-2 & 1 \\ -1-t^2 & -t-2 \end{bmatrix} x$$

(a) Show that the time varying state transformation

$$T(t)z(t) = \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix} z(t) = x(t)$$

leads to a time invariant set of dynamics $\dot{z} = Mz$ for an appropriate constant matrix M. Specify M.

(b) From the results of part (a), find z(1) and compute z(t) given that x(1) = $[1 \quad 1]^T$. Now determine x(t).

25. (a) Given a nonsingular state transformation Tz = x and the state dynamics $\dot{x} = Ax + Bu$, find $\hat{A}$ and $\hat{B}$ for $\dot{z} = \hat{A}z + \hat{B}u$ in terms of A, B, and T.
(b) If

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

find T so that

$$\hat{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

where λ_1 and λ_2 are the eigenvalues of A, i.e., the roots of the characteristic polynomial of A. Hint: answer is not unique.

REALIZATION

1. The zero-state response of a single-input single-output linear time invariant state model to the input $u(t) = 2\, 1^+(t)$ is

$$y(t) = 2\left(10e^{-t} - 6e^{t} - 4e^{-2t}\right)1^+(t) + 4 \times 1^+(t)$$

(a) Find the impulse response H(t).
(b) Find a state space realization of this system: **(i)** find D and B, **(ii)** find the characteristic equation and A, **(iii)** find C.

2. The step response of a particular linear time invariant system is

$$y(t) = \begin{bmatrix} 2e^{-2t} + 2 \\ 2e^{-2t} + te^{-2t} \end{bmatrix} 1^+(t)$$

(a) Find the impulse response of the system, simplified.
(b) Find the controllable canonical state model realization of the system.

Solution 2. (a)

$$H(t) = \frac{d}{dt}\left(\begin{bmatrix} 2e^{-2t} + 2 \\ 2e^{-2t} + te^{-2t} \end{bmatrix} 1^+(t)\right) = \begin{bmatrix} -4e^{-2t} \\ -3e^{-2t} - 2te^{-2t} \end{bmatrix} 1^+(t) + \begin{bmatrix} 4 \\ 2 \end{bmatrix} \delta(t) = Ce^{At}B1^+(t) + D\delta(t)$$

(b) By inspection $D = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Further

$$CB = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_2 \\ c_4 \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$$

and

$$CAB = \begin{bmatrix} c_1 & -4 \\ c_3 & -3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 & -4 \\ c_3 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} c_1 + 16 \\ c_3 + 12 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

Therefore

$$C = \begin{bmatrix} -8 & -4 \\ -8 & -3 \end{bmatrix}$$

3. The step response of a particular linear time invariant system is

$$y(t) = \left(be^{at} - ae^{bt} + a\right)1^+(t)$$

(a) Find the impulse response of the system.
(b) Find the controllable canonical state model realization of the system.

Check: $C = \begin{bmatrix} a^2b - ab^2 & 0 \end{bmatrix}$.

4. The zero-state step response of a single-input two-output linear time invariant state model is

$$y(t) = \begin{bmatrix} (10-6t)e^{-t} - 4e^{-2t} \\ -10e^{-t} + 4e^{-2t} \end{bmatrix}1^+(t) + \begin{bmatrix} 4 \\ -2 \end{bmatrix}1^+(t)$$

(a) Find the impulse response H(t).
(b) Find a state space realization of this system: **(i)** find D and B, **(ii)** find the characteristic equation and A, **(iii)** find C.

ANSWER: $C = \begin{bmatrix} -12 & -26 & -8 \\ 12 & 14 & 2 \end{bmatrix}$

5. The impulse response of a particular linear time invariant system is

$$h(t) = \begin{bmatrix} 1 & (t+1)e^t \end{bmatrix}1^+(t)$$

(a) (4 pts) Write down the GENERAL form of the impulse response in terms of the state model matrices A, B, C, and D.
(b) (16 pts) Find a state model realization of the system assuming A is in the controllable canonical form and $C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$.

Check: $B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 3 \end{bmatrix}$.

POLE PLACEMENT

1. (Pole placement) The state dynamics for a physical process are given by

$$\dot{x} = \begin{bmatrix} -a_4 & -a_3 & -a_2 & -a_1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

whose A-matrix is said to be in the so-called top companion form.
(a) Find the characteristic polynomial of A.

(b) Find state feedback, u <-- Fx + u, with F = [f_4, f_3, f_2, f_1] such that new system has natural frequencies at -1, -2, -3, and -4. Your answer should be in terms of the a_i.

2. (Pole placement*) A state model is given by

$$\dot{x}(t) = \left[\begin{array}{cc|cc} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] x(t) + \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} u(t)$$

Use the techniques developed in class and on the HW to find state feedback u <-- Fx + u so that (A +BF) is block diagonal, {-2, -3} is assigned to the first block, and {-1, -3} is assigned to the second block. Show the final F in matrix form.

3. (Pole placement) For the state dynamics,

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ a_1 & a_2 & a_3 \\ 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$

(a) Find the characteristic polynomial of the A-matrix.

(b) If $a_1 = a_2 = a_3 = 0$, find state feedback, u <--- Fx + u = $[f_1, f_2, f_3]$ x + u, such that the eigenvalues of the new system are all at -10.

4. (Pole placement in Luenberger form*[1]) (a) Find state feedback to assign the eigenvalues -1 ± j, -2 ± j to the Luenberger form below of the state model of a physical process. It is required that the resultant system have a block diagonal A-matrix.

$$\dot{x}(t) = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u(t)$$

(b) As a comparison, use the place-command in MATLAB's Control Systems Toolbox to assign the desired eigenvalues. Note that F = place(A,-B,eig) computes state feedback that assigns the eigenvalues stored in the array eig to the matrix A +BF.

5. Consider the state dynamics $\dot{x} = Ax + Bu$ where

$$A = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -a_2 & -a_1 & b_1 & b_2 \\ \hline 0 & 0 & 0 & 1 \\ d_1 & d_2 & -c_2 & -c_1 \end{array}\right]; B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Find F in terms of a_i, b_i, c_i, d_i, α_i, β_i, so that (i) the resulting system is block diagonal, and (ii) the polynomial of the system is $\pi_{A+BF}(\lambda) = (\lambda^2 + \alpha_1 \lambda + \alpha_2)(\lambda^2 + \beta_1 \lambda + \beta_2)$.

6. (Pole placement) Consider the state dynamics $\dot{x}$ = A x + B u where

$$A = \left[\begin{array}{cc|cc} -a_2 & -a_1 & b_1 & b_2 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ d_1 & d_2 & -c_2 & -c_1 \end{array}\right]; B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Find F in terms of a_i, b_i, c_i, d_i, α_i, β_i, so that (i) the resulting system is block diagonal, and (ii) the characteristic polynomial of the system is $\pi_{A+BF}(\lambda) = (\lambda^2 + \alpha_1 \lambda + \alpha_2)(\lambda^2 + \beta_1 \lambda + \beta_2)$.

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[1] See David G. Luenberger, "Canonical Forms for Linear Multivariable Systems," IEEE Trans. Auto Control, June 1967, pp. 290-293.

7. (MATLAB problem) An F-100 jet engine linearized about a specific operating point[1] has an A-matrix given by

A = [-3.245, -2.158, -915.5, 0.5731, 134.2;
1.642, -5.941, -281.6, 0.1897, 57.05;
1.685, -0.02554, -10.03, 0.007994, 0.5807;
0, 0, 0, -10, 0;
-2.163, 6.862, 740.5, 1.195, -171.5]

and a B-matrix given by

B = [0.01432, -355.3,-99.06, -15.49, 22.2e3;
0.2871, 728.6, 25.14, -64.87, 8.122e3;
-25.69, -103, 0.6333, -0.3213, -74.18;
10, 0, 0, 0, 0;
-0.1311, 329.5, -25, 62.57,-64.45e3]

(a) Find the eigenvalues of the system using the eig command in MATLAB.
(b) Find the characteristic polynomial of A using the poly command in MATLAB.
(c) If x(0) = $[1 \quad 1 \quad 1 \quad 1 \quad 1]^T \times 10^6$, find x(2) using the "expm" command in MATLAB assuming the input is zero.
(d) If x(1) = $[1 \quad 1 \quad 1 \quad 1 \quad 1]^T \times 10^{-6}$, find x(0) using the "expm" command in MATLAB assuming the input is zero. Does this operation appear to be well conditioned?
(e) The place command, F = place(A, -B,L), computes state feedback F to assign the eigenvalues listed in the array L to the matrix, A + BF. Use the place command to assign the eigenvalues {-10+j1, -10-j1,-2,-4,-10} to the system. Check that the eigenvalues of A+BF are in the required locations.
(f) Repeat part (e) using only the first two columns of B as the input matrix.

8. Consider the state dynamics

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

(a) Assuming a nonsingular state transformation, T(t) z(t) = x(t), find the equivalent state model in the z-coordintes.
(b) Suppose

$$A(t) = \begin{bmatrix} 1 & 1+\cos(t)-\sin(t) \\ 0 & 0 \end{bmatrix}, \text{and } B(t) = \begin{bmatrix} \sin(t) \\ 1 \end{bmatrix}$$

Show that the state transformation

[1] See Alternatives for Linear Multivariable Control by M. K. Sain, J. L. Peczkowski, and J. Melsa, National Engineering Consortium, Inc., Chicago, 1978, p. 24.

$$T(t)z(t) = \begin{bmatrix} 1 & \sin(t) \\ 0 & 1 \end{bmatrix} z(t) = x(t)$$

produces the time invariant state dynamics

$$\dot{z}(t) = \hat{A}z(t) + \hat{B}u(t)$$

(c) In the z-coordinates, find state feedback $\hat{F}$ to assign the eigenvalues {-1, -1} to the feedback system again in the z-coordinates.
(d) What is the form of this feedback in the original x-coordinates? (Hint: it should be time-varying; this illustrates how state transformations can be used to obtain feedback using time-invariant methods.)

9. Suppose you want the characteristic polynomial of the feedback system

$$\dot{x} = (A + BF)x$$

to be $\pi_{A+BF}(\lambda) = (\lambda^2 + 2\lambda + 10)(\lambda + 1)$ where

$$A = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ a & b & c \\ \hline c & b & a \end{array}\right], B = \left[\begin{array}{c|c} 0 & 0 \\ 1 & 0 \\ \hline 0 & 1 \end{array}\right]$$

(a) Write down the structure of F.
(b) Calculate the entries of F.

10. Consider the state dynamics $\dot{x} = Ax + Bu$ where

$$A = \left[\begin{array}{cc|cc} 0 & 1 & b_1 & b_2 \\ -a_2 & -a_1 & b_1 & b_2 \\ \hline 0 & 0 & 1 & 1 \\ d_1 & d_2 & -c_2 & -c_1 \end{array}\right]; B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Find F in terms of a_i, b_i, c_i, d_i, α_i, β_i, so that (i) the resulting system is block upper-TRIANGULAR, and (ii) the characteristic polynomial of the system is $\pi_{A+BF}(\lambda) = (\lambda^2 + \alpha_1 \lambda + \alpha_2)(\lambda^2 + \beta_1 \lambda + \beta_2)$.

11. Consider the state dynamics $\dot{x} = Ax + Bu$ where

$$A = \left[\begin{array}{cc|cc} -a_2 & -a_1 & b_1 & b_2 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ d_1 & d_2 & -c_2 & -c_1 \end{array}\right]; B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Find F in terms of a_i, b_i, c_i, d_i, α_i, β_i, so that (i) the resulting system is block diagonal, and (ii) the characteristic polynomial of the system is $\pi_{A+BF}(\lambda) = (\lambda^2 + \alpha_1 \lambda + \alpha_2)(\lambda^2 + \beta_1 \lambda + \beta_2)$.