Ascoli-Arzela

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$C[0,1]$ is a complete metric space with norm $||f|| := \sup |f|$ and metric $d(f,g) := ||f-g||$.

(a) Let $A \subseteq C[0,1]$ be closed, bounded, and equicontinuous; then we show that A is compact.

Prework:

I have a picture of what it means for this space to be bounded (the functions are contained in a tube). But what does it mean to be closed in a space like this? Well, essentially, that the boundary of the tube is contained as well.

Pf:

By the remark after 2.41, we shall instead show that $\forall E \subseteq A, E$ infinite, E has a limit point.

Since E is infinite, it has a further countable subset $E' \subseteq E$.

Since it's countable, let $E' = \{f_n\}$. It's pointwise bounded since bounded on $C[0,1]$ implies uniformly bounded imply pointwise bounded, and $\{f_n\}$ is equicontinuous, and let $K = [0,1]$ and note that this is compact over $\Re$. Now apply 7.25.

We have f_{n_k} a uniformly convergent subsequence, let f be its limit.

Since E is closed, $f \in E$ and thus E has a limit point.

$\square$

(b) Find $A \ni$ it's not compact but closed and bounded.

Following the hint, let $A = \{f_n(x)\} = \{\frac{1}{nx+1}\}$ (f_n defined on $[0,1]$).

Using basic calculus, we can see that it's uniformly bounded in $[0,1]$ as well.

However:

$|\frac{1}{nx+1} - \frac{1}{mx+1}|$

$= |\frac{(m-n)x}{nmx^2+(n+m)x+1}|$

Fix n, then by using some more basic calculus (when I say that, I mean, using derivatives to single out the critical points, etc.), we see that the minimum of the above is (assuming $n \neq m$:

$|\frac{x}{(n^2+n)x^2+(2n+1)x+1}|$

Which —IN THE SUP-NORM— implies that $\forall n, f_n$ is isolated. Therefore, A is trivially closed.

However, since f_n is isolated, $\exists$ infinite subset with no limit point.