# ECE 302

Midterm Examination 2

Summer 2016

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Instructions:

1. Do not open your exam booklet until the BEGIN signal is given.
2. Enter your name, ID #, and signature on the space provided below. Your signature indicates that you have not received unauthorized aid on the exam.
3. The exam is closed book and closed notes. Calculators are allowed.
4. An equation sheet is provided on the last page of the exam booklet.
5. There are 8 multiple choice problems (5 points and 10 points). You must CLEARLY indicate your answer to receive credit. No partial credit will be given for these problems.
6. There are 2 work-out problems (20 points each). You must show work to receive any credit. Partial credit will be given at the discretion of the instructor. Clearly designate final answers.
7. You have 60 minutes to complete this exam. When the end of the exam is announced, you must stop writing immediately. Anyone caught writing after the exam is over will get a grade of zero.

Name: **SOLUTION**

ID #:

Signature:

## Multiple Choice Section

**Questions 1 - 8 are multiple choice problems. Questions 1 - 4 are worth 5 points and questions 5 - 8 are worth 10 points. You must clearly select a single answer for each problem to receive credit. No partial credit will be given for these problems.**

1. (5 points) Let $X$ a random variable with characteristic function $\varphi_X(\omega)$. What is the characteristic function of $Y = X - a$?

   $\boxed{\text{(A)}}$ $\varphi_X(\omega)e^{-ja\omega}$

   (B) $\varphi_X(\omega)e^{ja\omega}$

   (C) $\varphi_X(\omega) - a$

   (D) $\varphi_X(\omega) + a$

   (E) $\varphi_X(\omega) - \varphi_X(a)$

   (F) $\varphi_X(\omega) + \varphi_X(a)$

   (G) $\varphi_X(a\omega)$

   (H) $\varphi_X(-a\omega)$

2. (5 points) An unfair coin with probability $p = 2/3$ of heads coming up is flipped. The mean of a random variable $X$ given heads comes up is $2/3$. The mean of $X$ is 1. What is the mean of $X$ given tails comes up?

   (A) 1/3

   (B) 4/9

   (C) 1/2

   (D) 5/9

   (E) 2/3

   (F) 1

   (G) 4/3

   $\boxed{\text{(H)}}$ 5/3

3. (5 points) Let $X$ and $Y$ be random variables with joint probability density function:

$$f_{X,Y}(x,y) = \begin{cases} 1 & , 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & , \text{else} \end{cases}$$

Which of the following expressions gives $\Pr(Y > X^2)$?

(A) $\int_0^1 \int_0^{x^2} dydx$

[(B)] $\int_0^1 \int_{x^2}^1 dydx$

(C) $\int_0^{x^2} \int_0^1 dxdy$

(D) $\int_{x^2}^1 \int_0^1 dxdy$

(E) None of the above.

4. (5 points) Let X be a Gaussian random variable with pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi 4}} \exp\left(\frac{-(x+7)^2}{8}\right).$$

What is $\Pr(X \leq 1)$?

(A) $\Phi(-3/2)$

(B) $\Phi(3/2)$

(C) $\Phi(-2)$

(D) $\Phi(2)$

(E) $\Phi(-3)$

(F) $\Phi(3)$

(G) $\Phi(-4)$

[(H)] $\Phi(4)$

5. (10 points) Let $X$ be a continuous random variable with cdf $F_X(x)$. Which of the following expressions is equal to $F_X(x|X \leq x+1)$ for all $x \in \mathbb{R}$?

(A) $F_X(x)$

(B) $F_X(x+1)$

(C) $F_X(x)F_X(x+1)$

(D) $F_X(x)/F_X(x+1)$

(E) $F_X(x+1)/F_X(x)$

(F) 0

(G) 1

6. (10 points) Let $X$ be a uniform random variable on the interval [1, 3]. What is $\mathbb{E}[1/X^2]$?

(A) 3/26

(B) 3/13

(C) 1/4

(D) 1/3

(E) 2/3

(F) 1

(G) 4/3

(H) 8/3

7. (10 points) Let $X$ and $Y$ be independent jointly Gaussian random variables with joint probability density function:

$$f_{X,Y}(x,y) = \frac{1}{2\pi 9}\exp\left(\frac{-(x^2+y^2+2x-4y+5)}{18}\right).$$

What is the $\mathbb{E}[X^2]$?

(A) 1
(B) 2
(C) 3
(D) 4
(E) 5
(F) 8
(G) 9
[(H)] 10

8. (10 points) Let $X$ and $Y$ be random variables with mean $\mu_X = 0$ and $\mu_Y > 0$, respectively, and $\text{Cov}[X,Y] < 0$. Which of the following statements is (are) **always** true?

I. $X$ and $Y$ are orthogonal.
II. $\mathbb{E}[XY] < 0$.
III. $\mathbb{E}[X(Y-\mu_Y)] < 0$.

(A) Only I.
(B) Only II.
(C) Only III.
(D) Only I and II.
(E) Only I and III.
[(F)] Only II and III.
(G) I, II, and III.
(H) None of the statements are true.

## Work-out Section

**Questions 9 and 10 are work-out problems (20 points each). You must show work to receive credit.**

9. Let X and Y be a random variables with

$$f_{X,Y}(x,y) = \begin{cases} c & , 0 \le x \le y \le 1 \\ 0 & , \text{else} \end{cases}$$

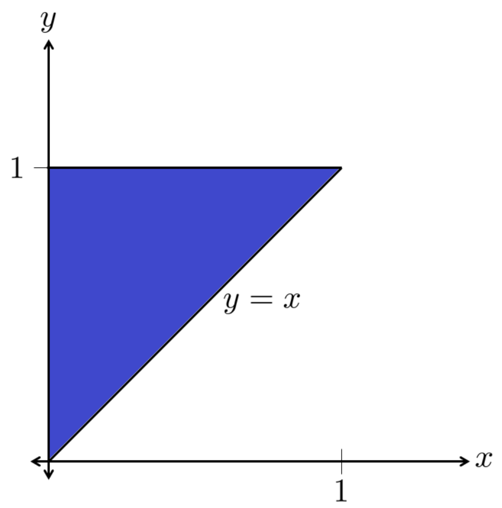
where $c$ is a constant.

(a) (4 points) Find $c$.

(b) (8 points) Find $\text{Cov}[X, Y]$.

(c) (8 points) Find $f_{Y|X}(y|x)$.

**Solution:**



(a)

$$
\begin{aligned}
1 &= \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} f_{X,Y}(x,y)\,dxdy \\
&= \int\limits_{0}^{1} \int\limits_{0}^{y} c\,dxdy \\
&= \int\limits_{0}^{1} cy\,dy \\
&= \frac{c}{2} \\
&\implies \boxed{c = 2}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\mathrm{Cov}[X,Y] &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \\
\mathbb{E}[XY] &= \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} xy f_{X,Y}(x,y)\,dxdy \\
&= \int\limits_{0}^{1} \int\limits_{0}^{y} 2xy\,dxdy \\
&= \int\limits_{0}^{1} y^3\,dy \\
&= \frac{1}{4} \\
\mathbb{E}[X] &= \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} x f_{X,Y}(x,y)\,dxdy \\
&= \int\limits_{0}^{1} \int\limits_{0}^{y} 2x\,dxdy \\
&= \int\limits_{0}^{1} y^2\,dy \\
&= \frac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[Y] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{X,Y}(x,y)\,dxdy \\
&= \int_{0}^{1}\int_{0}^{y} 2y\,dxdy \\
&= \int_{0}^{1} 2y^2\,dy \\
&= \frac{2}{3} \\
&\implies \boxed{\mathrm{Cov}[X,Y] = \tfrac{1}{36}}
\end{aligned}
$$

(c)

$$
\begin{aligned}
f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\
f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy \\
&= \int_{x}^{1} 2\,dy \\
&= 2(1-x)\,,\, 0 \le x \le 1 \\
&\implies f_{Y|X}(y|x) = \boxed{(1-x)^{-1},\ 0 \le x \le y < 1}
\end{aligned}
$$

Note that $f_{Y|X}(y|x)$ is a constant pdf with respect to $y$, but is parametrized by the value of $x$.

10. Let $X$ and $Y$ be random variables with pdf

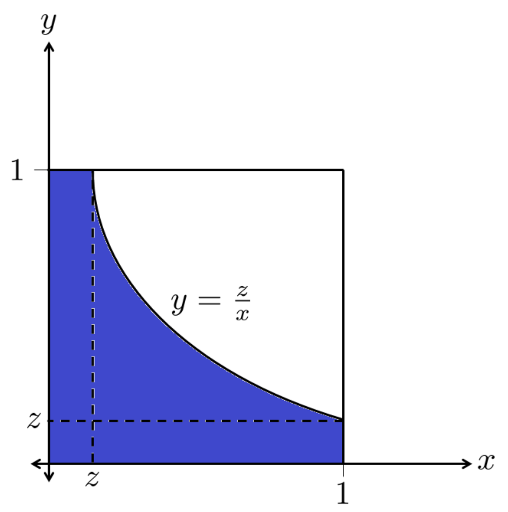
$$f_{X,Y}(x,y) = \begin{cases} 1 & , 0 \le x \le 1, 0 \le y \le 1 \\ 0 & , \text{else} \end{cases}$$

Let $Z = XY$.

(a) (14 points) Find the pdf of $Z$.

(b) (6 points) Find the variance of $Z$.

**Solution:**




(a) Distribution Method:

$$\begin{aligned} F_Z(z) &= \Pr(Z \le z) \\ &= \Pr(XY \le z) \\ &= \Pr(Y \le z/X) \end{aligned}$$

$$\int_0^z \int_0^1 dydx + \int_z^1 \int_0^{z/x} dydx$$

$$z + \int_z^1 \frac{z}{x}\,dx$$

$$z - z\ln(z),\ 0 \le z \le 1$$

$$\begin{aligned} f_Z(z) &= \frac{d}{dz}F_Z(z) \\ &= \frac{d}{dz}(z - z\ln(z)) \\ &= \boxed{-\ln(z),\ 0 < z \le 1} \end{aligned}$$

Density Method:
Let $W = Y$.

$$f_{Z,W}(z,w) = f_{Z,W}(x(z,w), y(z,w))\left|\frac{\partial(z,w)}{\partial(x,y)}\right|^{-1}$$

$$z = xy,\ w = y \implies x = z/w,\ y = w$$

$$\begin{aligned} \frac{\partial(z,w)}{\partial(x,y)} &= \left|\begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix}\right| \\ &= \left|\begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix}\right| \\ &= y \end{aligned}$$

$$\begin{aligned} \implies f_{Z,W}(z,w) &= f_{X,Y}(\frac{z}{w}, w)\frac{1}{w} \\ &= \frac{1}{w}, \quad 0 \le \frac{z}{w} \le 1, 0 < w \le 1 \end{aligned}$$

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_{Z,W}(z,w)dw \\ &= \int_z^1 \frac{1}{w}\,dw \\ &= \boxed{-\ln(z),\ 0 < z \le 1} \end{aligned}$$

Since $0 \le z \le 1$, $-\ln(z) \ge 0$, so $f_Z(z)$ is a valid pdf.

(b)

$$
\begin{aligned}
\mathrm{Var}[Z] &= \mathbb{E}[Z^2] - (\mathbb{E}[Z])^2 \\
&= \mathbb{E}[X^2Y^2] - (\mathbb{E}[XY])^2 \\
\mathbb{E}[X^2Y^2] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2y^2 f_{X,Y}(x,y)\, dxdy \\
&= \int_{0}^{1}\int_{0}^{1} x^2y^2\, dxdy \\
&= \frac{1}{9} \\
\mathbb{E}[XY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\, dxdy \\
&= \int_{0}^{1}\int_{0}^{1} xy\, dxdy \\
&= \frac{1}{4} \\
\implies & \boxed{\mathrm{Var}[Z] = \tfrac{1}{9} - \tfrac{1}{16} = \tfrac{7}{144}}
\end{aligned}
$$

**Equations**

$$F_X(x|X\in A) = \frac{\int\limits_{-\infty}^{x} f_X(x')1_A(x')dx'}{\Pr(X\in A)}$$

$$f_X(x|X\in A) = \frac{f_X(x)1_A(x)}{\Pr(X\in A)}$$

$$\Phi(x) = \int\limits_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$\mathbb{E}[X^n] = \frac{1}{j^n}\frac{d^n}{d\omega^n}\varphi_X(\omega)\Big|_{\omega=0}$$

$$F_{U,V}(u,v) = \iint\limits_{(x,y):g(x,y)\le u,\, h(x,y)\le v} f_{X,Y}(x,y)\, dxdy$$

$$F_Z(z) = \iint\limits_{(x,y):g(x,y)\le z} f_{X,Y}(x,y)\, dxdy$$

$$f_{U,V}(u,v) = f_{X,Y}(x(u,v),y(u,v))\left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \left|\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}\right|$$

$$\frac{d}{dy}\int_{a(y)}^{b(y)} f(x)dx = f(b(y))\frac{db(y)}{dy} - f(a(y))\frac{da(y)}{dy}$$