EE 301 Exam 1 JVK

1 (a)

$$e^{j\omega_0 n} \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \sum_k h[k] e^{j\omega_0 (n-k)}$$

$$= e^{j\omega_0 n} \underbrace{\sum_k h[k] e^{-j\omega_0 k}}_{H(\omega_0)}$$

(b) $$E_\infty(x) = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$E_\infty(y) = \int_{-\infty}^{\infty} |x(at+b)|^2 \, dt$$

c.o.v.
$\tau = at + b$
$d\tau = |a| \, dt$

$$= \int_{-\infty}^{\infty} |x(\tau)|^2 \frac{d\tau}{|a|} \qquad (a \neq 0)$$

$$= \frac{1}{|a|} E_\infty(x) \quad \text{provided } a \neq 0.$$

(c)-i

$$y[n] = x[n] + 2x[n-1]$$

Linear can be directly verified. Time Invariant because diff. equation coeffs are constant. Clearly has memory. Clearly causal. Is BIBO stable. Say $|x[n]| \leq B \implies |y[n]| \leq 3B$.

(c)-ii

$$y[n] = x[n] + \tfrac{1}{2} x[n+1]$$

Still LTI with memory. BIBO stable holds. Now non-causal.

(c)-iii $\quad y[n] = \sum_{k=n-2}^{n+2} k\,x[k]$

Is linear. Has memory. Is non-causal.
Is time-varying (coefficient = time k).
Is not BIBO. For example take

$$x[k] = 1 \quad \forall k \qquad \text{(bounded input)}$$

get

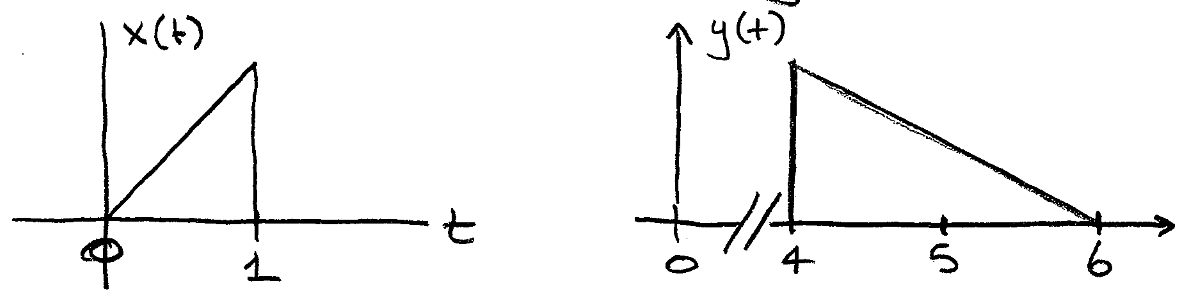
$$y[n] = n-2 + n-1 + n + n+1 + n+2$$
$$= 5n \longleftarrow \text{an unbounded output.}$$

(c)-iv. $\quad y[n] = e^{x[n]}$

Not linear. Is time-invariant. Is memoryless.
Is causal. Is BIBO.

$$\text{If } |x[n]| \le B \implies |y[n]| \le e^{B}$$

(d)



$$3 - \tfrac{1}{2}t = 0 \rightarrow t = 6$$
$$3 - \tfrac{1}{2}t = 1 \rightarrow t = 4$$

(e) $h[n] = \left(\frac{1}{2}\right)^n u[n] \qquad h_{inv}[n] = h_{inv}[0]\,\delta[n] + h_{inv}[1]\,\delta[n-1]$

To be inverse system must have

$$h * h_{inv}[n] = \delta[n] = \begin{cases} 1 & n=0 \\ 0 & n \neq 0. \end{cases}$$

$$h * h_{inv}[n] = h_{inv}[0] \cdot h[n] + h_{inv}[1]\, h[n-1]$$

$$h_{inv}[0] \cdot \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \cdots & h[n] \end{pmatrix}$$

$$+$$

$$h_{inv}[1] \begin{pmatrix} 0 & 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots & h[n-1] \end{pmatrix}$$

$$||$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & \delta[n] \end{pmatrix}.$$

Solving we pick

$$h_{inv}[0] = 1$$

$$h_{inv}[1] = -\frac{1}{2}$$

2. $y^{(2)} + y = e^{t}$ $\qquad$ $y^{(1)}(0) = 0$

$\qquad\qquad\qquad\qquad\qquad\qquad y^{(0)}(0) = 1.$

As in class char. equation is $s^2 + 1 = 0 \Longrightarrow s = \pm j$
The additional root for e^t is $s = +1$. No repeated roots so

$$y(t) = c_1 \cos t + c_2 \sin t + c_3 e^{t}$$

Plugging into ODE

$$c_3 e^{t} + c_3 e^{t} = e^{t} \Longrightarrow c_3 = 1/2$$

$$y(t) = c_1 \cos t + c_2 \sin t + \tfrac{1}{2} e^{t}$$

$$y^{(1)}(t) = -c_1 \sin t + c_2 \cos t + \tfrac{1}{2} e^{t}$$

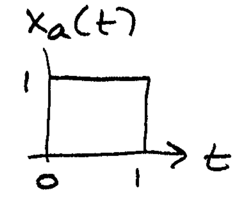
$$y(0) = 1 = c_1 + \tfrac{1}{2} \Longrightarrow c_1 = 1/2$$

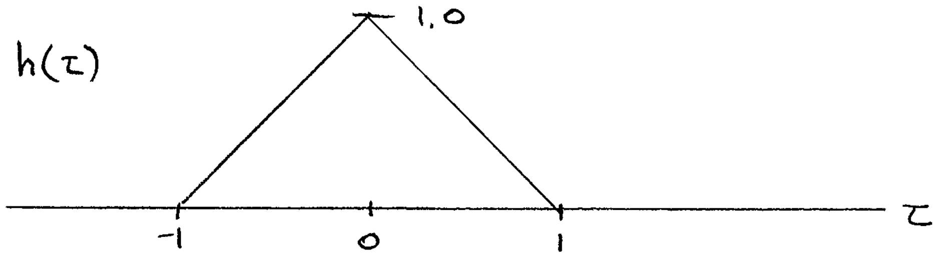
$$y^{(1)}(0) = 0 = c_2 + \tfrac{1}{2} \Longrightarrow c_2 = -1/2$$

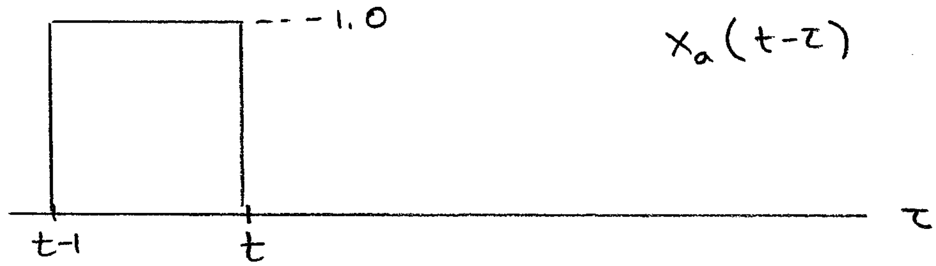
$$\therefore \; y(t) = \tfrac{1}{2}\left(\cos t - \sin t + e^{t}\right) \quad t \geq 0.$$

3.

(a) Calculate $y_a(t) = x_a * h(t)$ where
Draw picture to see the cases:







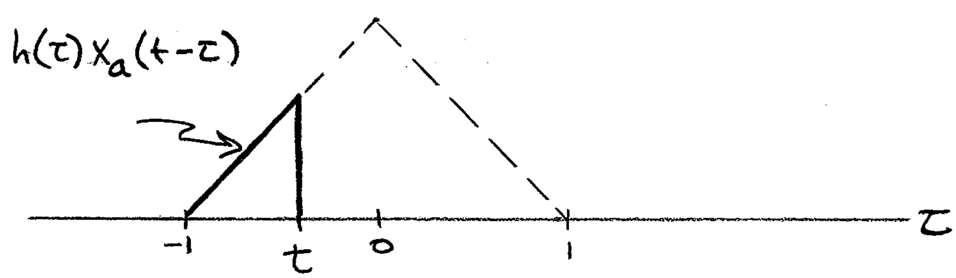
The cases to consider are:

(i) $t < -1 \implies y_a(t) = 0$

(ii) $t-1 > 1 \iff t > 2 \implies y_a(t) = 0$

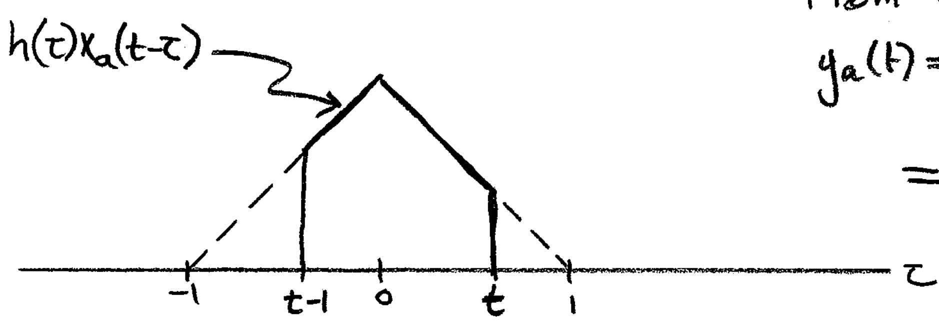
(iii) $-1 < t < 0$

In this case the integrand in $y_a(t) = \int h(\tau)\, x_a(t-\tau)\, d\tau$ looks like



From triangle area formula $y_a(t) = \frac{1}{2}(t+1)^2$

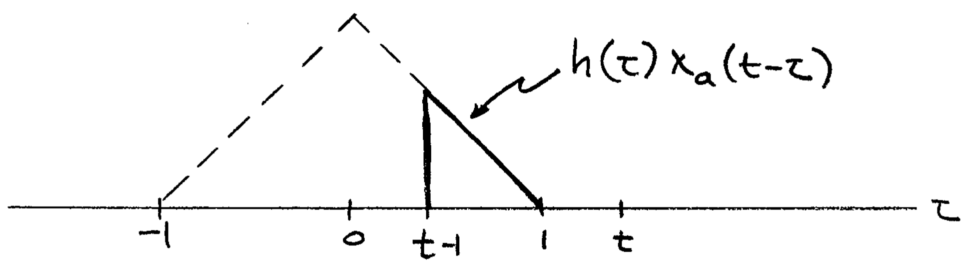
(iv) $0 < t < 1$ so integrand is



From triangle formula

$$
\begin{aligned}
y_a(t) &= 1 - \tfrac{1}{2}\left(t-1-(-1)\right)^2 - \tfrac{1}{2}(1-t)^2 \\
&= 1 - \tfrac{1}{2}t^2 - \tfrac{1}{2}(t-1)^2 \\
&= 1 - \tfrac{1}{2}t^2 - \tfrac{1}{2}t^2 + t - \tfrac{1}{2} \\
&= -t^2 + t + \tfrac{1}{2}
\end{aligned}
$$

(v) $0 < t-1 < 1 \iff 1 < t < 2$ so integrand is:

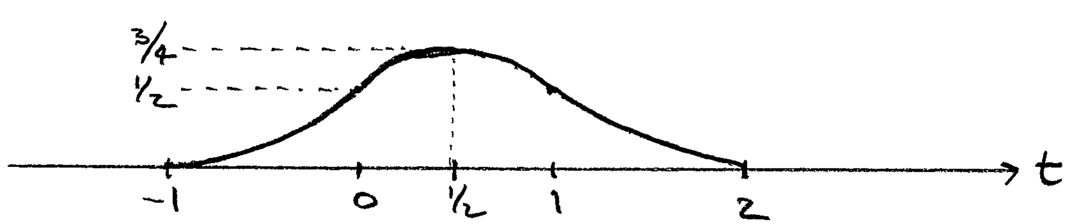


From triangle formula

$$y_a(t) = \frac{1}{2}(1-t+1)^2 = \frac{1}{2}(t-2)^2$$

Summarizing

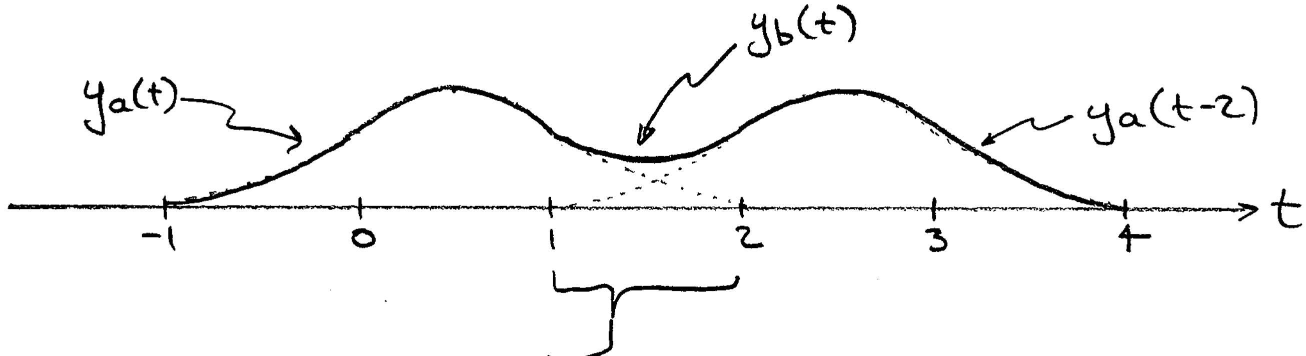
$$y_a(t) = \begin{cases} 0 & \text{if} \quad t < -1 \text{ or } t > 2 \\ \frac{1}{2}(t+1)^2 & \text{if} \quad -1 \le t \le 0 \\ -t^2 + t + \frac{1}{2} & \text{if} \quad 0 \le t \le 1 \\ \frac{1}{2}(t-2)^2 & \text{if} \quad 1 \le t \le 2 \end{cases}$$



(b) $x_b(t) = x_a(t) + x_a(t-2)$

$\Longrightarrow$ LTI properties imply that

$$y_b(t) = y_a(t) + y_a(t-2)$$



The equation for $y_b(t)$ valid over the interval $1<t<2$ would have to be

$$\tfrac{1}{2}(t-2)^2 + \tfrac{1}{2}(t-2+1)^2$$

$$= \tfrac{1}{2}(t-2)^2 + \tfrac{1}{2}(t-1)^2$$

$$= t^2 - 3t + 5/2$$

$$= t^2 - 3t + \tfrac{9}{4} + \tfrac{5}{2} - \tfrac{9}{4}$$

$$= \left(t - \tfrac{3}{2}\right)^2 + \tfrac{1}{4}$$

(c) Here we note that

$$x_c(t) = -\frac{1}{2} + \sum_{k=-\infty}^{\infty} x_a(t-2k)$$

$\therefore$ LTI properties tell us that

$$y_c(t) = \underbrace{-\frac{1}{2} * h(t)}_{} + \sum_{k=-\infty}^{\infty} y_a(t-2k)$$

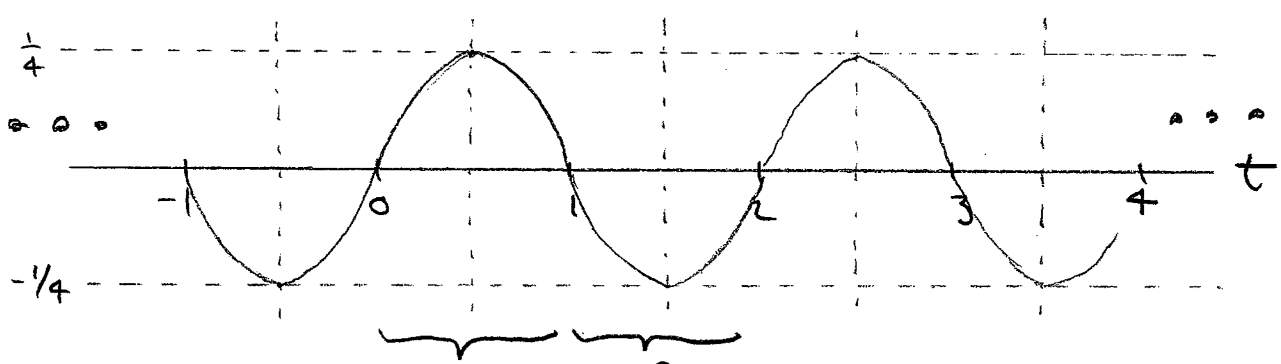
this is the only part we don't know

$$-\frac{1}{2} * h(t) = \int_{-\infty}^{\infty} h(\tau)\left(-\frac{1}{2}\right) d\tau$$

$$= -\frac{1}{2} \cdot \text{area under } h(t)$$

$$= -\frac{1}{2} \cdot 1 = -\frac{1}{2}$$

$$\therefore y_c(t) = -\frac{1}{2} + \sum_{k=-\infty}^{\infty} y_a(t-2k)$$



pieces of parabolas knitted together (not a sine wave)