ECE 301

Division 2, Fall 2006
Instructor: Mimi Boutin
Midterm Examination 3

Instructions:

1. Wait for the "BEGIN" signal before opening this booklet. In the meantime, read the instructions below and fill out the requested info.
2. You have 50 minutes to answer the 5 questions contained in this exam. **When the end of the exam is announced, you must stop writing immediately**. Anyone caught writing after the exam is over will get a grade of zero.
3. This booklet contains 14 pages. The last five pages contain a table of formulas and properties. You may tear out these pages **once the exam begins**. Each transform and each property is labeled with a number. To save time, you may use these numbers to specify which transform/property you are using when justifying your answer. In general, if you use a fact which is *not* contained in this table, you must explain why it is true in order to get full credit. The only exceptions are the properties of the ROC, which can be used without justification.
4. This is a closed book exam. The only personal items allowed are pens/pencils, erasers and something to drink. Scratch paper will be provided by the exam supervisors. Anything else is strictly forbidden.

Name:______________________________________

Email:______________________________________

Signature:____________________________________

1. Consider the following CT signals:

$$\begin{aligned} x(t) &= \frac{\sin(5\pi t)}{\pi t}, \\ y(t) &= x(t)e^{j\omega_1 t}, \\ z(t) &= x(t)\cos(\omega_2 t)\,. \end{aligned}$$

(5 pts) a) Is the signal $x(t)$ band limited? (Answer yes/no and justify.) If you answered yes, what is the signal's Nyquist rate?

(5 pts) b) Assuming that $\omega_1 > 0$, is the signal $y(t)$ band limited? (Answer yes/no and justify.) If you answered yes, what is the signal's Nyquist rate?

(5 pts) c) Assuming that $\omega_2 > 0$, is the signal $z(t)$ band limited? (Answer yes/no and justify.) If you answered yes, what is the signal's Nyquist rate?

(10 pts) d) Can one recover x(t) from y(t)? (Answer yes/no. If you answer yes, explain how. If you answered no, explain why not.)

(15 pts) e) Can one recover x(t) from z(t)? (Answer yes/no. If you answer yes, explain how. If you answered no, explain why not.)

(15 pts) f) Impulse-train sampling is used to obtain

$$x_p[n] = \sum_{k=-\infty}^{\infty} x(n)\delta(n - kN).$$

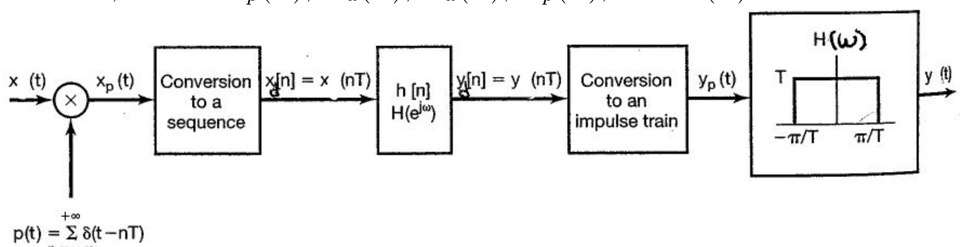
If the sampling period is $N = \frac{2}{11}$, will aliasing occur? Justify your answer.

(15 pts) g) Impulse-train sampling is used to obtain

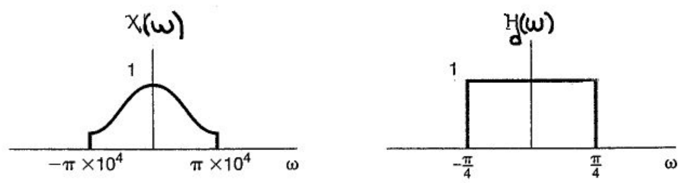
$$y_p[n] = \sum_{k=-\infty}^{\infty} y(n)\delta(n - kN).$$

If $\omega_1 = -3\pi$ and the sampling period is $N = \frac{1}{6}$, will aliasing occur? Justify your answer.

(20 pts) **2.** The following figure shows the overall system for filtering a CT signal using a DT filter. If $\mathcal{X}(\omega)$ and $\mathcal{H}_d(\omega)$ are as shown below, with $\frac{1}{T} = 20$kHZ, sketch $\mathcal{X}_p(\omega)$, $\mathcal{X}_d(\omega)$, $\mathcal{Y}_d(\omega)$, $\mathcal{Y}_p(\omega)$, and $\mathcal{Y}(\omega)$.



(a)



(b)

(15 pts) **3.** Using the definition of the Laplace transform (i.e. do not simply take the answer from the table), compute the Laplace transform of

$$x(t) = e^{-5t}u(-t)$$

(20 pts) **4.** Using the definition of the Laplace transform (i.e. do not simply take the answer from the table), compute the inverse z-transform of

$$X(z) = \frac{1+z}{1+\frac{1}{3}z}, |z| > 3.$$

(15 pts)**5.** A discrete-time LTI system as unit impulse response

$$h[n] = -n3^n u[-n-1].$$

What is the output of the system when the input is

$$x[n] = (1+j)^n?$$

(Leave your answer in unsimplified form.)

Table

Fourier Series of CT Periodic Signals with period T

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t} \tag{1}$$

$$a_k = \frac{1}{T}\int_0^T x(t) e^{-jk\left(\frac{2\pi}{T}\right)t} dt \tag{2}$$

Some CT Fourier series

Signal	a_k	
$e^{j\omega_0 t}$	$a_1 = 1, a_k = 0$ else.	(3)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0 & T_1 < \lvert t\rvert < \frac{T}{2} \end{cases}$ and $x(t+T) = x(t)$	$\dfrac{\sin k\omega_0 T_1}{k\pi}$	(4)
$\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$a_k = \dfrac{1}{T}$	(5)

Fourier Series of DT Periodic Signals with period N

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \tag{6}$$

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n} \tag{7}$$

CT Fourier Transform

$$\begin{aligned} \text{F.T. : } \mathcal{X}(\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt && (8) \\ \text{Inverse F.T.: } x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega && (9) \end{aligned}$$

Properties of CT Fourier Transform

Let $x(t)$ be a continuous-time signal and denote by $\mathcal{X}(\omega)$ its Fourier transform. Let $y(t)$ be another continuous-time signal and denote by $\mathcal{Y}(\omega)$ its Fourier transform.

	Signal	F.T.	
Linearity:	$ax(t) + by(t)$	$a\mathcal{X}(\omega) + b\mathcal{Y}(\omega)$	(10)
Time Shifting:	$x(t - t_0)$	$e^{-j\omega t_0}\mathcal{X}(\omega)$	(11)
Frequency Shifting:	$e^{j\omega_0 t}x(t)$	$\mathcal{X}(\omega - \omega_0)$	(12)
Time and Frequency Scaling:	$x(at)$	$\frac{1}{\lvert a\rvert}\mathcal{X}\left(\frac{\omega}{a}\right)$	(13)
Multiplication:	$x(t)y(t)$	$\frac{1}{2\pi}\mathcal{X}(\omega) * \mathcal{Y}(\omega)$	(14)
Convolution:	$x(t) * y(t)$	$\mathcal{X}(\omega)\mathcal{Y}(\omega)$	(15)
Differentiation in Time:	$\frac{d}{dt}x(t)$	$j\omega\mathcal{X}(\omega)$	(16)

Some CT Fourier Transform Pairs

$$\begin{aligned} e^{j\omega_0 t} &\xrightarrow{\mathcal{F}} 2\pi\delta(\omega - \omega_0) && (17) \\ 1 &\xrightarrow{\mathcal{F}} 2\pi\delta(\omega) && (18) \\ \frac{\sin Wt}{\pi t} &\xrightarrow{\mathcal{F}} u(\omega + W) - u(\omega - W) && (19) \\ \delta(t) &\xrightarrow{\mathcal{F}} 1 && (20) \\ u(t) &\xrightarrow{\mathcal{F}} \frac{1}{j\omega} + \pi\delta(\omega) && (21) \\ \sum_{n=-\infty}^{\infty} \delta(t - nT) &\xrightarrow{\mathcal{F}} \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{T}) && (22) \end{aligned}$$

DT Fourier Transform

$$\text{F.T.:}\mathcal{X}(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad (23)$$

$$\text{Inverse F.T.: } x[n] = \frac{1}{2\pi}\int_{2\pi} \mathcal{X}(\omega)e^{j\omega n}d\omega \quad (24)$$

Properties of DT Fourier Transform

Let $x(t)$ be a signal and denote by $\mathcal{X}(\omega)$ its Fourier transform. Let $y(t)$ be another signal and denote by $\mathcal{Y}(\omega)$ its Fourier transform.

	Signal	F.T.	
Linearity:	$ax[n] + by[n]$	$a\mathcal{X}(\omega) + b\mathcal{Y}(\omega)$	(25)
Time Shifting:	$x[n-n_0]$	$e^{-j\omega n_0}\mathcal{X}(\omega)$	(26)
Frequency Shifting:	$e^{j\omega_0 n}x[n]$	$\mathcal{X}(\omega-\omega_0)$	(27)
Time Reversal:	$x[-n]$	$\mathcal{X}(-\omega)$	(28)
Time Exp.:	$x_k[n] = \begin{cases} x[\frac{n}{k}], & \text{if } k \text{ divides } n \\ 0, & \text{else.} \end{cases}$	$\mathcal{X}(\omega)$	(29)
Multiplication:	$x[n]y[n]$	$\frac{1}{2\pi}\mathcal{X}(\omega) * \mathcal{Y}(\omega)$	(30)
Convolution:	$x[n] * y[n]$	$\mathcal{X}(\omega)\mathcal{Y}(\omega)$	(31)
Differencing in Time:	$x[n] - x[n-1]$	$(1-e^{-j\omega})\mathcal{X}(\omega)$	(32)

Some DT Fourier Transform Pairs

$$\sum_{k=0}^{N-1} a_k e^{jk(\frac{2\pi}{N})n} \xrightarrow{\mathcal{F}} 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - \frac{2\pi k}{N}) \quad (33)$$

$$1 \xrightarrow{\mathcal{F}} 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l) \quad (34)$$

$$(35)$$

$$\delta[n] \xrightarrow{\mathcal{F}} 1 \quad (36)$$

$$u[n] \xrightarrow{\mathcal{F}} \frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \quad (37)$$

Laplace Transform

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{38}$$

Properties of Laplace Transform

Let $x(t)$, $x_1(t)$ and $x_2(t)$ be three CT signals and denote by $X(s)$, $X_1(s)$ and $X_2(s)$ their respective Laplace transform. Let R be the ROC of $X(s)$, let R_1 be the ROC of $X_1(z)$ and let R_2 be the ROC of $X_2(s)$.

	Signal	L.T.	ROC	
Linearity:	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$	(39)
Time Shifting:	$x(t - t_0)$	$e^{-st_0}X(s)$	R	(40)
Shifting in s:	$e^{s_0 t}x(t)$	$X(s - s_0)$	$R + s_0$	(41)
Conjugation:	$x^*(t)$	$X^*(s^*)$	R	(42)
Time Scaling:	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{s}{a}\right)$	aR	(43)
Convolution:	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$	(44)
Differentiation in Time:	$\frac{d}{dt}x(t)$	$sX(s)$	At least R	(45)
Differentiation in s:	$-tx(t)$	$\frac{dX(s)}{ds}$	R	(46)
Integration :	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	At least $R \cap \mathcal{R}e\{s\} > 0$	(47)

Some Laplace Transform Pairs

Signal	LT	ROC	
$u(t)$	$\frac{1}{s}$	$\mathcal{R}e\{s\} > 0$	(48)
$-u(-t)$	$\frac{1}{s}$	$\mathcal{R}e\{s\} < 0$	(49)
$u(t)\cos(\omega_0 t)$	$\frac{s}{s^2 + \omega_0^2}$	$\mathcal{R}e\{s\} > 0$	(50)
$e^{-\alpha t}u(t)$	$\frac{1}{s + \alpha}$	$\mathcal{R}e\{s\} > -\alpha$	(51)
$-e^{-\alpha t}u(-t)$	$\frac{1}{s + \alpha}$	$\mathcal{R}e\{s\} < -\alpha$	(52)
$\delta(t)$	1	all s	(53)

z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \tag{54}$$

Properties of z-Transform

Let $x[n]$, $x_1[n]$ and $x_2[n]$ be three DT signals and denote by $X(z)$, $X_1(z)$ and $X_2(z)$ their respective z-transform. Let R be the ROC of $X(z)$, let R_1 be the ROC of $X_1(z)$ and let R_2 be the ROC of $X_2(z)$.

	Signal	z-T.	ROC	
Linearity:	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least $R_1 \cap R_2$	(55)
Time Shifting:	$x[n - n_0]$	$z^{-n_0}X(z)$	$R,$ but perhaps adding/deleting $z = 0$	(56)
Time Shifting:	$x[-n]$	$X(z^{-1})$	R^{-1}	(57)
Scaling in z:	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R	(58)
Conjugation:	$x^*(t)$	$X^*(z^*)$	R	(59)
Convolution:	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least $R_1 \cap R_2$	(60)

Some z-Transform Pairs

Signal	LT	ROC	
$u[n]$	$\frac{1}{1 - z^{-1}}$	$\lvert z\rvert > 1$	(61)
$-u(-n-1)$	$\frac{1}{1 - z^{-1}}$	$\lvert z\rvert < 1$	(62)
$\alpha^n u[n]$	$\frac{1}{1 - \alpha z^{-1}}$	$\lvert z\rvert > \alpha$	(63)
$-\alpha^n u[-n-1]$	$\frac{1}{1 - \alpha z^{-1}}$	$\lvert z\rvert < \alpha$	(64)
$n\alpha^n u[n]$	$\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$	$\lvert z\rvert > \alpha$	(65)
$-n\alpha^n u[-n-1]$	$\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$	$\lvert z\rvert < \alpha$	(66)
$\delta[n]$	1	all z	(67)