Midterm Examination 2
ECE 301
Division 1, Fall 2008

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Instructions:

1. Wait for the "BEGIN" signal before opening this booklet. In the meantime, read the instructions below and fill out the requested info.
2. You have 50 minutes to complete the 5 questions contained in this exam. **When the end of the exam is announced, you must stop writing immediately**. Anyone caught writing after the exam is over will get a grade of zero.
3. This booklet contains 10 pages. The last four pages contain a table of formulas and properties. You may tear out these pages **once the exam begins**. TABLE USE RULES: You may use any fact contained in the table without justification. Simply write the number of the corresponding table item to indicate which fact you are using from the table. If you use a non-trivial fact that is *not* contained in the table, you must justify (i.e., prove) it in order to get full credit.
4. This is a closed book exam. Calculators, cell phones, and i-pods are strictly forbidden.

Name:____________________________________

Email:____________________________________

Signature:________________________________

Itemized Scores

Problem 1:

Problem 2:

Problem 3:

Problem 4:

Problem 5:

Total:

(15 pts) **1.** Using the definition of the Fourier transform (*not* the table of Fourier transform pairs and *not* the table of FT properties), compute the Fourier transform of the DT signal:

$$x[n] = \left(\frac{1}{1+2j} \right)^{n+1} u[n+1].$$

(Simplify your answer as much as possible.)

(15 pts) **2.** Consider the CT the signal $x(t)$ periodic with period $T = 4$ defined by

$$x(t) = \begin{cases} 0, & -2 < t < -1 \\ 1, & -1 \leq t \leq 1 \\ 0, & 1 < t \leq 2 \end{cases} .$$

Compute the Fourier transform of x(t). (Simplify your answer as much as possible.)

(10 pts) **3.** Demonstrate the Frequency Shifting property of the DT fourier transform (Property 36 in the table).

(15 pts) **4.** A DT LTI system has frequency response $H(e^{j\omega}) = \frac{1}{1+\frac{1}{2}e^{j\omega}}$. Compute the system's response to the input $x[n] = 2^n u[-n]$.

(15 pts) **5.** Consider the LTI system defined by the differential equation

$$y''(t) + 4y'(t) + 3y(t) = x'(t) + 2x(t).$$

What is the unit impulse response of this system? (Justify your answer.)

Table

1 Definition of the Continuous-time Fourier Transform

Let $x(t)$ be a signal and denote by $\mathcal{X}(\omega)$ its Fourier transform.

$$\text{Fourier Transform: } \mathcal{X}(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \quad (1)$$

$$\text{Inverse Fourier Transform: } x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{X}(\omega)e^{j\omega t}d\omega \quad (2)$$

2 Some Continuous-time Fourier Transforms

$$e^{j\omega_0 t} \xrightarrow{\mathcal{F}} 2\pi\delta(\omega - \omega_0) \quad (3)$$

$$1 \xrightarrow{\mathcal{F}} 2\pi\delta(\omega) \quad (4)$$

$$\frac{\sin Wt}{\pi t} \xrightarrow{\mathcal{F}} u(\omega + W) - u(\omega - W) \quad (5)$$

$$\delta(t) \xrightarrow{\mathcal{F}} 1 \quad (6)$$

$$u(t) \xrightarrow{\mathcal{F}} \frac{1}{j\omega} + \pi\delta(\omega) \quad (7)$$

$$e^{-at}u(t), \mathcal{R}e\{a\} > 0 \xrightarrow{\mathcal{F}} \frac{1}{a + j\omega} \quad (8)$$

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) \xrightarrow{\mathcal{F}} \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{T}) \quad (9)$$

3 Properties of the Continuous-time Fourier Transform

Let $x(t)$ be a continuous-time signal and denote by $\mathcal{X}(\omega)$ its Fourier transform. Let $y(t)$ be another continuous-time signal and denote by $\mathcal{Y}(\omega)$ its Fourier transform.

$$\text{Linearity: } ax(t)+by(t) \xrightarrow{\mathcal{F}} a\mathcal{X}(\omega)+b\mathcal{Y}(\omega) \tag{10}$$

$$\text{Time Shifting: } x(t-t_0) \xrightarrow{\mathcal{F}} e^{-j\omega t_0}\mathcal{X}(\omega) \tag{11}$$

$$\text{Frequency Shifting: } e^{j\omega_0 t}x(t) \xrightarrow{\mathcal{F}} \mathcal{X}(\omega-\omega_0) \tag{12}$$

$$\text{Conjugation: } x^*(t) \xrightarrow{\mathcal{F}} \mathcal{X}^*(-\omega) \tag{13}$$

$$\text{Scaling: } x(at) \xrightarrow{\mathcal{F}} \frac{1}{|a|}\mathcal{X}\left(\frac{\omega}{a}\right) \tag{14}$$

$$\text{Multiplication: } x(t)y(t) \xrightarrow{\mathcal{F}} \frac{1}{2\pi}\mathcal{X}(\omega)*\mathcal{Y}(\omega) \tag{15}$$

$$\text{Convolution: } x(t)*y(t) \xrightarrow{\mathcal{F}} \mathcal{X}(\omega)\mathcal{Y}(\omega) \tag{16}$$

$$\text{Differentiation in Time: } \frac{d}{dt}x(t) \xrightarrow{\mathcal{F}} j\omega\mathcal{X}(\omega) \tag{17}$$

$$x(t)\text{real} \xrightarrow{\mathcal{F}} \mathcal{X}(\omega)=\mathcal{X}^*(-\omega) \tag{18}$$

$$x(t)\text{real and even} \xrightarrow{\mathcal{F}} \mathcal{X}(\omega)\text{real and even} \tag{19}$$

$$x(t)\text{real and odd} \xrightarrow{\mathcal{F}} \mathcal{X}(\omega)\text{pure imaginary and odd} \tag{20}$$

$$\text{Parseval's Relation: } \int_{-\infty}^{\infty}|x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|\mathcal{X}(\omega)|^2\,d\omega \tag{21}$$

4 Fourier Series of Continuous-time Periodic Signals with period T

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t} \tag{22}$$

$$a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\left(\frac{2\pi}{T}\right)t}dt \tag{23}$$

5 Fourier Series of Discrete-time Periodic Signals with period N

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \tag{24}$$

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n} \tag{25}$$

6 Definition of the Discrete-time Fourier Transform

Let $x[n]$ be a discrete-time signal and denote by $\mathcal{X}(\omega)$ its Fourier transform.

$$\text{Fourier Transform: } \mathcal{X}(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \tag{26}$$

$$\text{Inverse Fourier Transform: } x[n] = \frac{1}{2\pi} \int_{2\pi} \mathcal{X}(\omega) e^{j\omega n} d\omega \tag{27}$$

7 Some Discrete-time Fourier Transforms

$$\sum_{k=0}^{N-1} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \xrightarrow{\mathcal{F}} 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - \frac{2\pi k}{N}) \tag{28}$$

$$1 \xrightarrow{\mathcal{F}} 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l) \tag{29}$$

$$\frac{\sin Wn}{\pi n}, 0 < W < \pi \xrightarrow{\mathcal{F}} \mathcal{X}(\omega) = \begin{cases} 1, & 0 \le |\omega| < W \\ 0, & \pi \ge |\omega| > W \end{cases} \tag{30}$$

$$\mathcal{X}(\omega)\text{periodic with period } 2\pi$$

$$\delta[n] \xrightarrow{\mathcal{F}} 1 \tag{31}$$

$$\alpha^n u[n], |\alpha| < 1 \xrightarrow{\mathcal{F}} \frac{1}{1 - \alpha e^{-j\omega}} \tag{32}$$

$$(n+1)\alpha^n u[n], |\alpha| < 1 \xrightarrow{\mathcal{F}} \frac{1}{(1 - \alpha e^{-j\omega})^2} \tag{33}$$

8 Properties of the Discrete-time Fourier Transform

Let $x[n]$ and $y[n]$ be DT signals. Denote by $\mathcal{X}(\omega)$ and $\mathcal{Y}(\omega)$ their Fourier transforms.

$$\text{Linearity: } ax[n] + by[n] \xrightarrow{\mathcal{F}} a\mathcal{X}(\omega) + b\mathcal{Y}(\omega) \quad (34)$$

$$\text{Time Shifting: } x[n-n_0] \xrightarrow{\mathcal{F}} e^{-j\omega n_0}\mathcal{X}(\omega) \quad (35)$$

$$\text{Frequency Shifting: } e^{j\omega_0 n}x[n] \xrightarrow{\mathcal{F}} \mathcal{X}(\omega-\omega_0) \quad (36)$$

$$\text{Conjugation: } x^*[n] \xrightarrow{\mathcal{F}} \mathcal{X}^*(-\omega) \quad (37)$$

$$\text{Time Reversal: } x[-n] \xrightarrow{\mathcal{F}} \mathcal{X}(-\omega) \quad (38)$$

$$x_k[n] = \begin{cases} x[\frac{n}{k}], & \text{if } k \text{ divides } n \\ 0, & \text{else.} \end{cases} \xrightarrow{\mathcal{F}} \mathcal{X}(\omega) \quad (39)$$

$$\text{Multiplication: } x[n]y[n] \xrightarrow{\mathcal{F}} \frac{1}{2\pi}\mathcal{X}(\omega) * \mathcal{Y}(\omega) \quad (40)$$

$$\text{Convolution: } x(t) * y(t) \xrightarrow{\mathcal{F}} \mathcal{X}(\omega)\mathcal{Y}(\omega) \quad (41)$$

$$\text{Differentiation: } x[n] - x[n-1] \xrightarrow{\mathcal{F}} (1-e^{-j\omega})\mathcal{X}(\omega) \quad (42)$$

$$\text{Accumulation: } \sum_{k=-\infty}^{n} x[k] \xrightarrow{\mathcal{F}} \frac{\mathcal{X}(\omega)}{1-e^{-j\omega}} + \pi\mathcal{X}(0)\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k) \quad (43)$$

$$x[n]\text{real} \xrightarrow{\mathcal{F}} \mathcal{X}(\omega) = \mathcal{X}^*(-\omega) \quad (44)$$

$$x[n]\text{real and even} \xrightarrow{\mathcal{F}} \mathcal{X}(\omega)\text{real and even} \quad (45)$$

$$x[n]\text{real and odd} \xrightarrow{\mathcal{F}} \mathcal{X}(\omega)\text{pure imaginary and odd} \quad (46)$$

$$\text{Parseval's Relation: } \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |\mathcal{X}(w)|^2\,d\omega \quad (47)$$