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(15 pts) 1. Compute the Fourier transform of the DT signal

$$x[n] = n^2 u[n-2] - n^2 u[n+2]$$

(Express your answer as a linear combination of sine and/or cosine functions.)

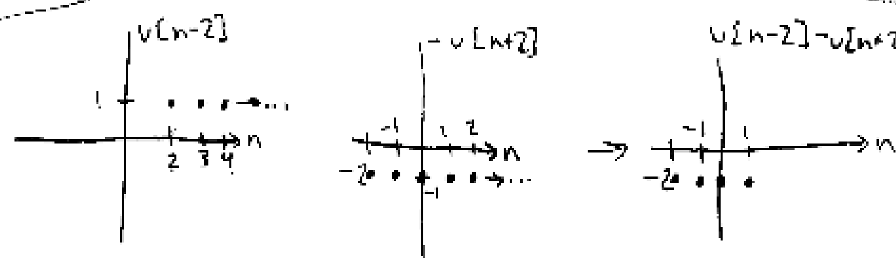
$$x[n] = n^2 (u[n-2] - u[n+2])$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} n^2 (u[n-2] - u[n+2]) e^{-j\omega n}$$

$$= \sum_{n=-2}^{1} -n^2 e^{-j\omega n}$$

$$= -4e^{j\omega 2} - e^{j\omega} - e^{-j\omega}$$

$$= -4e^{j\omega 2} - 2\cos(\omega)$$



(15 pts) 2. Show that the Fourier transform of the CT signal $x(t) = \cos(\omega_0 t)$ is $\mathcal{X}(\omega) = \pi\delta(\omega + \omega_0) - \pi\delta(\omega - \omega_0)$.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)\, e^{jwt}\, dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\pi\delta(w+w_1) + \pi\delta(w-w_0)\right) e^{jwt}\, dw$$

$$= \frac{1}{2}\left(\int_{-\infty}^{\infty} \delta(w+w_0)\, e^{jwt}\, dw + \int_{-\infty}^{\infty} \delta(w-w_1)\, e^{jwt}\, dw\right)$$

(By the sifting property of δ) $$= \frac{1}{2} e^{jw_0t} + \frac{1}{2} e^{-jw_0t} = \cos(w_0 t)$$

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$x[n]$ = real and even

(15 pts) 3. Given is a DT signal $x[n] = \frac{1}{g[n]^2}$ where $g[n]$ is a pure imaginary signal and an odd function of n.

a) Bob claims that the Fourier transform of $x[n]$ is $X(\omega) = \frac{j}{\cos\omega}$. Explain why Bob's answer is wrong.

$x[n]$ real and even $\xrightarrow{?}$ $X(w)$ real and even (35)

but $X(w)$ (Bob's) is imaginary $\rightarrow$ wrong.

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b) Alice says that the Fourier transform of $x[n]$ is $X(\omega) = \frac{1}{\sin\omega}$. Could Alice be right? Explain.

$x[n]$ real and even $\xrightarrow{?}$ $X(w)$ real and even (35)

but $X(w)$ (Alice's) is odd $\rightarrow$ wrong.

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c) Devin says that the Fourier transform of $x[n]$ is $X(\omega) = \frac{1}{\omega^2}$. Could Devin be right? Explain.

The Fourier transform of a DT signal is always periodic with period 2π.

but Devin's $X(w)$ is not $\rightarrow$ wrong.

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4. A discrete-time LTI system has frequency response

$$\mathcal{H}(w) = H(e^{jw}) = \frac{2}{1 - \frac{3}{4}e^{-jw} + \frac{1}{8}e^{-2jw}}.$$

(15 pts) a) Derive a difference equation relating the input and the output of this system. (Use the properties of the Fourier transform listed in the table to justify your answer.)

$$\mathcal{H}(w) = \frac{\mathcal{Y}(w)}{\mathcal{X}(w)} = \frac{2}{1 - \frac{3}{4}e^{-jw} + \frac{1}{8}e^{-2jw}}$$

$$\Rightarrow \left(1 - \frac{3}{4}e^{-jw} + \frac{1}{8}e^{-2jw}\right)\mathcal{Y}(w) = 2\mathcal{X}(w)$$

$$\Rightarrow \frac{1}{8}e^{-2jw}\mathcal{Y}(w) - \frac{3}{4}e^{-jw}\mathcal{Y}(w) + \mathcal{Y}(w) = 2\mathcal{X}(w)$$

$$\Rightarrow \mathcal{F}^{-1}\left(\frac{1}{8}e^{-2jw}\mathcal{Y}(w) - \frac{3}{4}e^{-jw}\mathcal{Y}(w) + \mathcal{Y}(w)\right) = \mathcal{F}^{-1}(2\mathcal{X}(w))$$

By (25) and (24): $\frac{1}{8}y[n-2] - \frac{3}{4}y[n-1] + y[n] = 2x[n]$

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(10 pts) b) What is the Fourier transform of the output when the input is $x[n] = \left(\frac{1}{4}\right)^n u[n]$?

$$\mathcal{X}(w) = \mathcal{F}(x[n]) = \mathcal{F}\left(\left(\tfrac{1}{4}\right)^n u[n]\right) = \frac{1}{1-\frac{1}{4}e^{-jw}} \quad (\text{By } 4.2)$$

$$\mathcal{Y}(w) = \mathcal{X}(w)\,\mathcal{H}(w) = \frac{1}{1-\frac{1}{4}e^{-jw}} \cdot \frac{2}{1-\frac{3}{4}e^{-jw}+\frac{1}{8}e^{-2jw}}$$

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(15 pts) b) Find the unit impulse response of this system.

$$\mathcal{H}(\omega) = \frac{2}{1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-2j\omega}} \times \frac{8}{8} = \frac{16}{(e^{-j\omega})^2-6e^{-j\omega}+8}$$

$$= \frac{16}{(e^{-j\omega}-2)(e^{-j\omega}-4)} = \frac{A}{e^{-j\omega}-2} + \frac{B}{e^{-j\omega}-4} = \frac{-8}{e^{-j\omega}-2} + \frac{8}{e^{-j\omega}-4}$$

$$= \frac{4}{1-\frac{1}{2}e^{-j\omega}} - \frac{2}{1-\frac{1}{4}e^{-j\omega}}$$

$$h[n] = \mathcal{F}^{-1}(\mathcal{H}(\omega)) = \mathcal{F}^{-1}\left(\frac{4}{1-\frac{1}{2}e^{-j\omega}} - \frac{2}{1-\frac{1}{4}e^{-j\omega}}\right)$$

By (4.2) and (LA) $= 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]$

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(20 pts) 5. Use the definition of the Fourier transform (*not* the properties listed in the table) to prove the following Fourier transform property.

$$x(at+b) \overset{F}{\longleftrightarrow} \frac{e^{jw\frac{b}{a}}}{-a} X\left(\frac{w}{a}\right) \text{ for any } a, b \text{ real numbers with } a < 0.$$

$$\mathcal{F}\big(x(at+b)\big) = \int_{-\infty}^{\infty} x(at+b)\, e^{-jwt}\, dt$$

Let $t' = at+b \Rightarrow t = \frac{1}{a}t' - \frac{b}{a} \rightarrow dt = \frac{1}{a}dt'$

$t'(\infty) = a \cdot \infty = -\infty \quad , \quad t'(-\infty) = a \cdot -\infty = \infty$

$$\Rightarrow = \int_{\infty}^{-\infty} x(t')\, e^{-jw\left(\frac{1}{a}t' - \frac{b}{a}\right)} \cdot \frac{1}{a}\, dt'$$

$$= \frac{e^{jw\frac{b}{a}}}{-a} \int_{-\infty}^{\infty} x(t')\, e^{-j\frac{w}{a}t'}\, dt'$$

Let $w' = \frac{w}{a}$

$$\Rightarrow = \frac{e^{jw\frac{b}{a}}}{-a} \int_{-\infty}^{\infty} x(t')\, e^{-jw't'}\, dt'$$

$$= \frac{e^{jw\frac{b}{a}}}{-a} X(w') = \frac{e^{jw\frac{b}{a}}}{-a} X\left(\frac{w}{a}\right)$$

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