6 May 2003 **Name:** Solution

EE 438 **Final Exam** **Spring 2003**

- You have 120 minutes to work the following five problems.
- Be sure to show all your work to obtain full credit.
- Unless explicitly stated to the contrary, you must simplify your answer as much as possible to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are permitted.

1. (25 pts.) Consider the DT system described by the following equation:

$$y[n] = x[2n]$$

For each property listed below, either prove that the property holds or provide a counter-example to demonstrate that it does not hold.

a. (5) linearity

b. (5) shift-invariance

c. (4) causality

d. (4) bounded-input-bounded-ouput stability

For the next two parts to this problem, we consider the "frequency" response of this system.

e. (4) Find the response $y_\omega[n]$ of this system to a complex exponential $x[n] = e^{j\omega n}$.

f. (3) For the input signal $x[n]$ in part e above, is it possible to find a function $H(\omega)$, such that $y_\omega[n] = H(\omega)x[n]$? Why or why not?

a. TRUE.

$$T[\alpha_1 x_1[n] + \alpha_2 x_2[n]] = \alpha_1 x_1[2n] + \alpha_2 x_2[2n] = \alpha_1 T[x_1[n]] + \alpha_2 T[x_2[n]]$$

b. FALSE.

Eg

$$x[n] \longrightarrow \boxed{T} \longrightarrow x[2n] \longrightarrow \boxed{\text{Delay by } n_0 \neq 0} \longrightarrow y_1[n] = x[2(n-n_0)] = x[2n-2n_0]$$

$$x[n] \longrightarrow \boxed{\text{Delay by } n_0} \longrightarrow x[n-n_0] \longrightarrow \boxed{T} \longrightarrow y_2[n] = x[2n-n_0]$$

$$y_1[n] \neq y_2[n]$$

1. (continued)

C. FALSE

$y[1]$ depends on $x[2]$ ← future value

d TRUE

Assume $|x[n]| \leq M$, M constant, $\forall n$

Then $|y[n]| = |x[2n]| \leq M$ $\forall n$

BIBO.

e

$$y_\omega[n] = x[2n] = e^{j2\omega n}$$

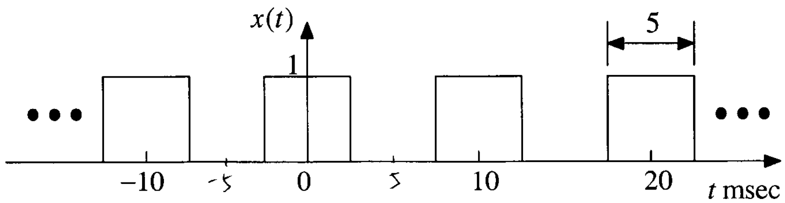
f

$$\frac{y_\omega[n]}{x[n]} = \frac{e^{j2\omega n}}{e^{j\omega n}} = e^{j\omega n} = H(n,\omega)$$

since we cannot get rid of n in $H(n,\omega)$,

it is NOT possible to find $H(\omega)$ — which should be a function only of ω

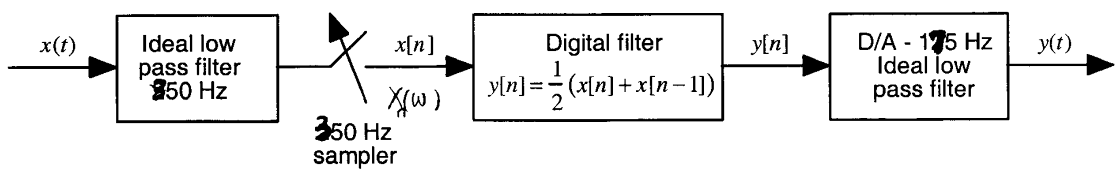
2. (25 pts) The signal $x(t)$ shown below



is bandlimited to 350 Hz with an ideal low pass filter, then sampled at a 350 Hz rate. The sampled signal is filtered with a digital filter described by the difference equation

$$y[n] = \frac{1}{2}(x[n] + x[n-1]).$$

Then the signal is passed through an ideal D/A convertor that can be modeled as a lowpass filter with cutoff frequency at 175 Hz. The entire system is shown below:



Find simple expressions for the following signals:

a. (10) $x[n]$

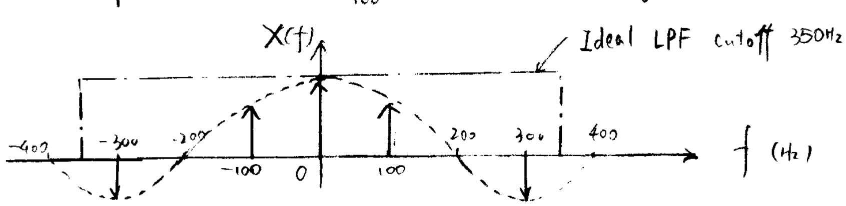
b. (5) $y[n]$

c. (10) $y(t)$

Hint: It's best to work this problem all the way through in the frequency domain, even though you are asked for time-domain results.

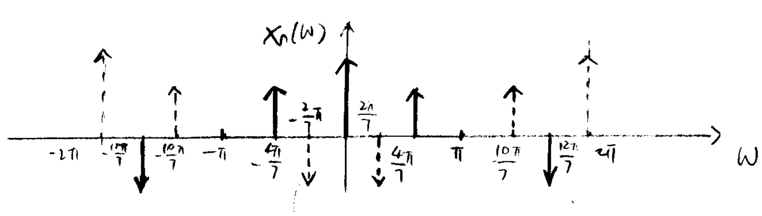
a. $x[t] = \text{rep}_{0.01}\left(\text{rect}\left(\frac{t}{0.005}\right)\right)$

$X(f) = 100\ \text{comb}_{100}\left(0.005\ \text{sinc}(0.005 f)\right)$



$\frac{300}{350} \times 2\pi = \frac{6}{7} \cdot 2\pi$

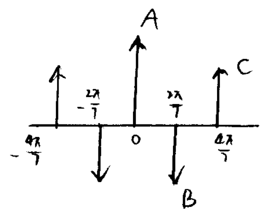
$\frac{100}{350} \times 2\pi = \frac{2}{7} \cdot 2\pi$



Aliasing Components

2. (continued)

From $-\pi$ to π, one entire period of $X_n(\omega)$ contains 5 $\delta(\cdot)$



Magnitudes:

$$A = 100 \times 0.005 = 0.5$$

$$B = 100 \times 0.005 \times \text{Sinc}(0.005 \times 300) = -0.106$$

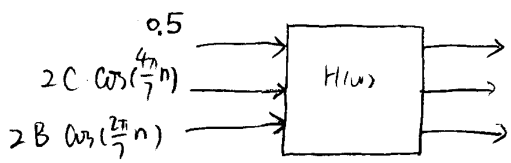
$$C = 100 \times 0.005 \times \text{Sinc}(0.005 \times 100) = 0.318$$

$$\therefore X_n(\omega) = 0.5\delta(\omega) + 0.318\left(\delta(\omega - \tfrac{4}{7}\pi) + \delta(\omega + \tfrac{4}{7}\pi)\right) - 0.106\left(\delta(\omega - \tfrac{2\pi}{7}) + \delta(\omega + \tfrac{2\pi}{7})\right)$$

$$\therefore x[n] = 0.5 + 0.637\cos\left(\tfrac{4}{7}\pi n\right) - 0.212\cos\left(\tfrac{2\pi}{7} n\right)$$

b. $y[n] = \frac{1}{2}(x[n] + x[n-1]) \Rightarrow Y(\omega) = \frac{1}{2}(1 + e^{-j\omega})X(\omega)$

$$\Rightarrow H(\omega) = \tfrac{1}{2}(1 + e^{-j\omega}) = e^{j\frac{\omega}{2}} \cdot \cos(\tfrac{\omega}{2})$$



$H(0) \cdot 0.5 = 0.5$

$\left|H(\tfrac{4\pi}{7})\right| \cdot 2C \cdot \cos(\tfrac{4\pi}{7}n + \tfrac{1}{2} \times \tfrac{4\pi}{7}) = 0.624 \cdot 2C \cdot \cos(\tfrac{4\pi}{7}n + \tfrac{2\pi}{7})$ $= 0.397\cos(\tfrac{4\pi}{7}n + \tfrac{2\pi}{7})$

$\left|H(\tfrac{2\pi}{7})\right| \cdot 2B \cos(\tfrac{2\pi n}{7} + \tfrac{1}{2} \times \tfrac{2\pi}{7}) = 0.901 \times 2B \times \cos(\tfrac{2\pi}{7}n + \tfrac{\pi}{7})$ $= -0.191 \cdot \cos(\tfrac{2\pi n}{7} + \tfrac{\pi}{7})$

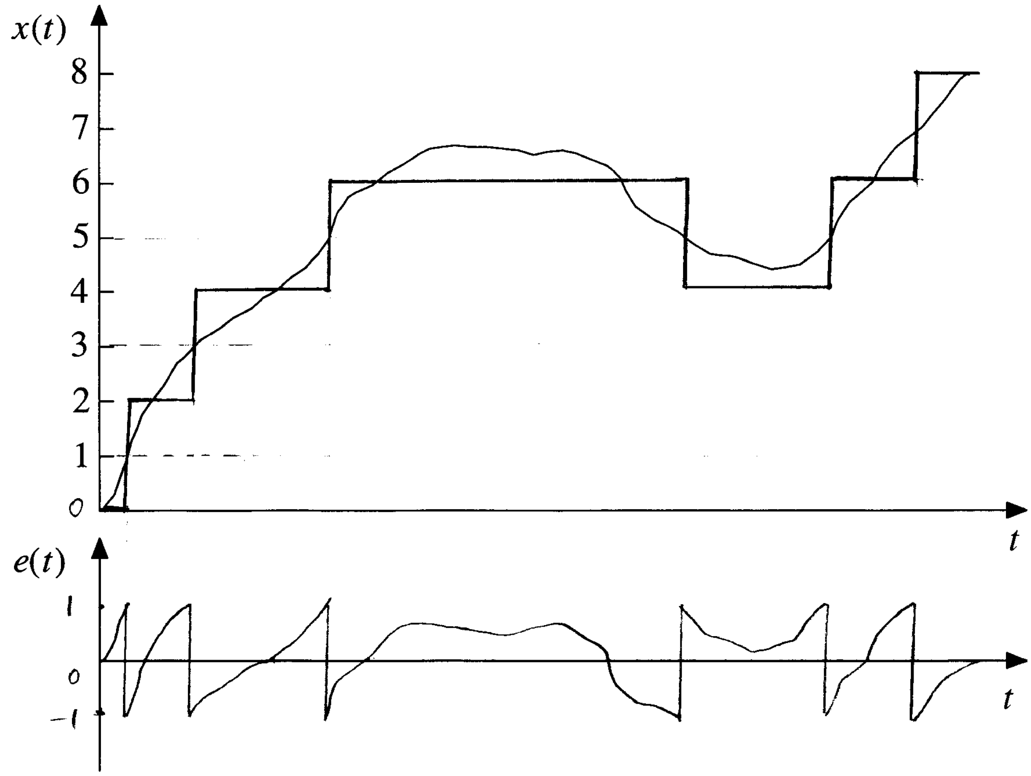
$$\therefore y[n] = 0.5 + 0.397\cos\left(\tfrac{4\pi n}{7} + \tfrac{2\pi}{7}\right) - 0.191\cos\left(\tfrac{2\pi n}{7} + \tfrac{\pi}{7}\right)$$

c. $t = nT_s = \frac{n}{350}$ $\therefore n = 350t$

$$\therefore y(t) = y[350t] = 0.5 + 0.397\cos\left(200\pi t + \tfrac{2\pi}{7}\right) - 0.191\cos\left(100\pi t + \tfrac{\pi}{7}\right)$$

3. (25 pts.)

Consider the signal $x(t)$ shown below. Assume that this signal is to be quantized with interval 2 between quantization output levels. with 0 as the first output level.



a. (6) Carefully draw the quantized signal $x_q(t)$ on the same axes as that of $x(t)$.

b. (5) Plot the error $e(t) = x(t) - x_q(t)$ on the axes indicated, below the plot for $x(t)$.

c. (7) Estimate the mean-squared value e_{ms} of this signal.

d. (7) Suppose that $x(t)$ is a random signal that obeys the density function

$$f_X(x) = \frac{1}{8} rect[(x-4)/8]$$

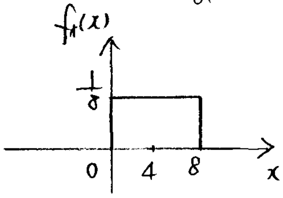
Find the signal to noise ratio at the output of the quantizer.

3. (continued)

c.

$$\left.\begin{array}{l} e_{ms} = \dfrac{\Delta^2}{12} \\ \text{noise} \overset{\text{Approx}}{\sim} \text{Uniform}\left[-\dfrac{\Delta}{2}, \dfrac{\Delta}{2}\right] \quad \therefore \quad \Delta = 2 \end{array}\right\} \Rightarrow e_{ms} = \frac{4}{12} = \frac{1}{3}$$

d.



$$S^2 = E[x^2(t)] = \int x^2 f_X(x)\,dx$$

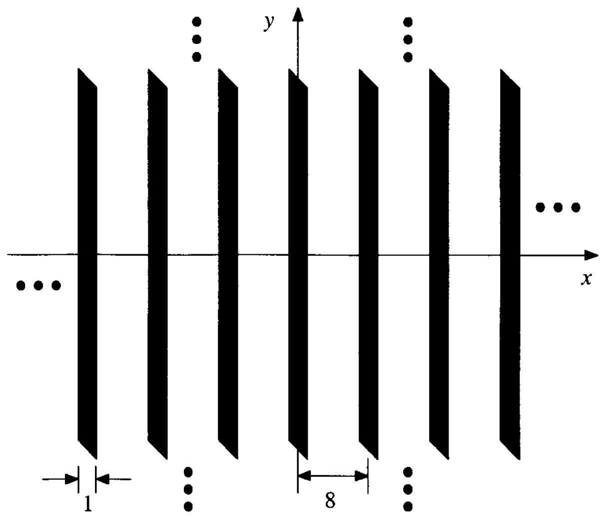
$$= \int_0^8 x^2 \frac{1}{8}\,dx = \frac{1}{8} \cdot \frac{1}{3} \cdot x^3 \Big|_0^8 = \frac{64}{3}$$

$$N^2 = e_{ms} = \frac{1}{3}$$

$$SNR = 10\log_{10}\left(\frac{S^2}{N^2}\right) = 10\log_{10}(64) = 18.06 \text{ dB}$$

4. (25 pts.)

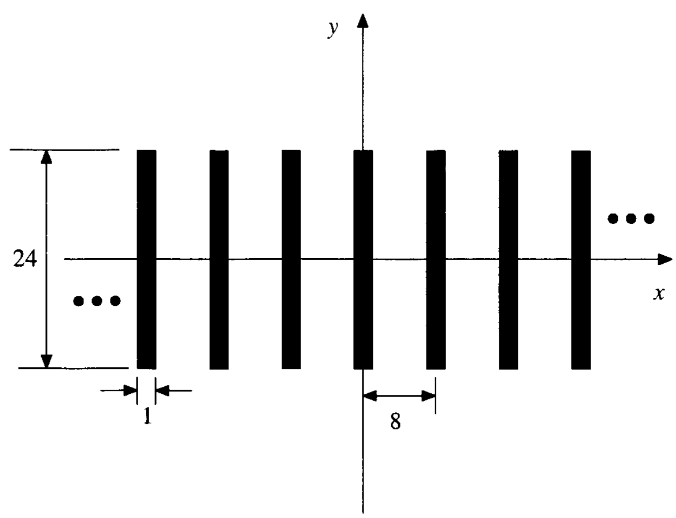
a. (15) Consider the signal $f(x,y)$ shown below



which has value 1 in the shaded areas, and value 0, elsewhere. The bars are infinitely long along the y axis; and there are infinitely many of them along the x axis.

Find a simple expression for the CSFT $F(u,v)$ of this signal, and sketch it.

b. (10) Consider the signal $g(x,y)$ shown below

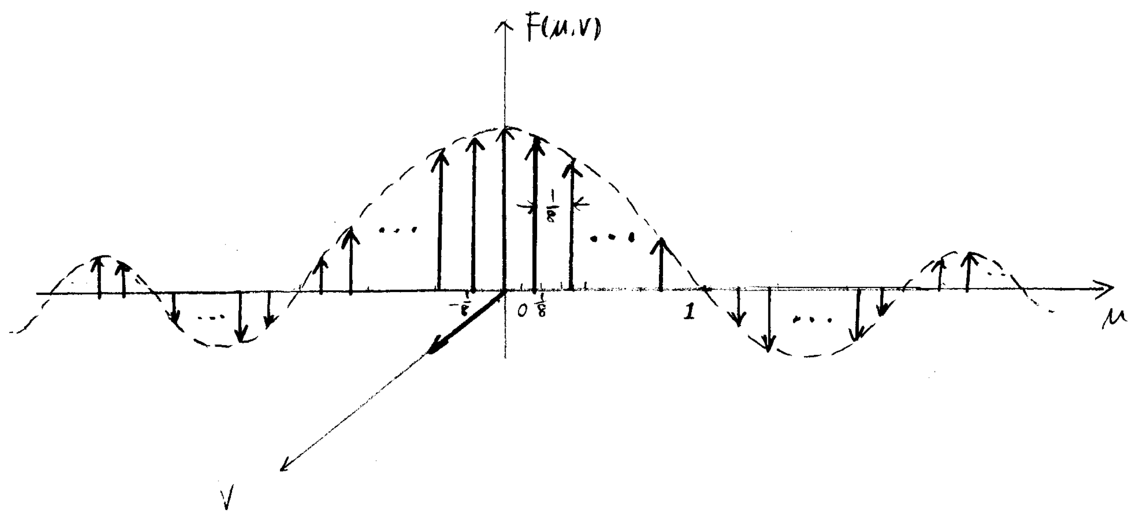


which has value 1 in the shaded areas, and value 0, elsewhere. Here bars have length 24 along the y axis; but there are still infinitely many of them along the x axis.

Find a simple expression for the CSFT $G(u,v)$ of this signal, and sketch it.

a. $f(x,y) = \text{rep}_8[\text{rect}(x)] \cdot 1$

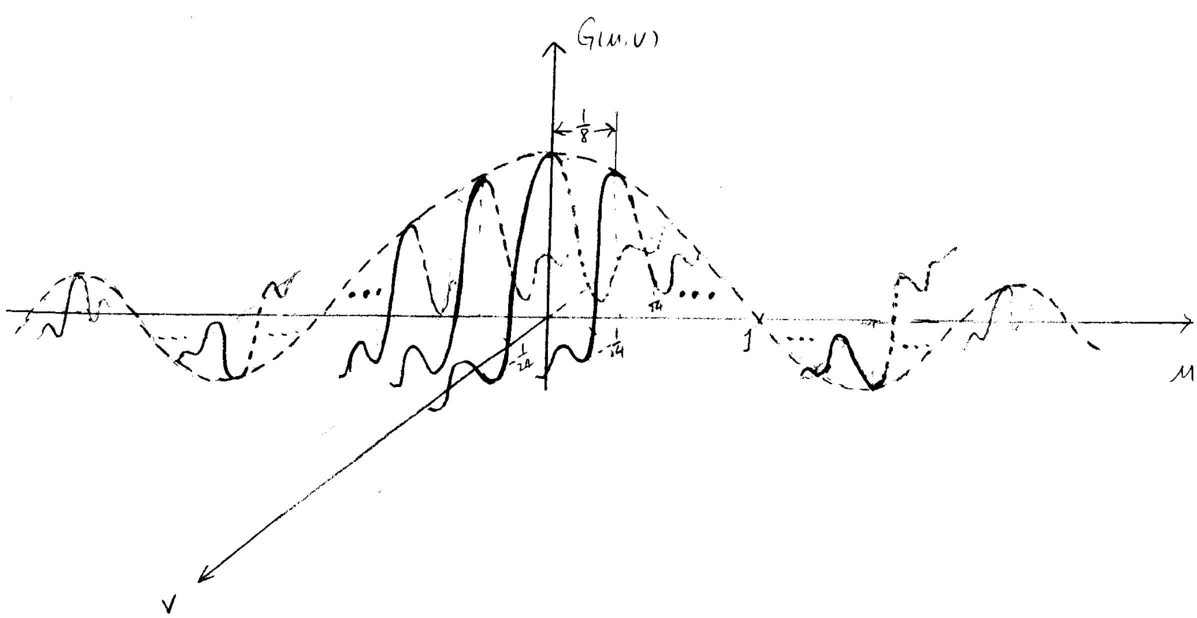
$F(\mu,v) = \frac{1}{8}\text{Comb}_{\frac{1}{8}}[\text{Sinc}(\mu)] \cdot \delta(v)$



b. $g(x,y) = f(x,y) \cdot \text{rect}(\frac{y}{24})$

$G(\mu,v) = \text{CSFT}(\text{rep}_8[\text{rect}(x)] \cdot \text{rect}(\frac{y}{24}))$

$= \frac{1}{8}\text{Comb}_{\frac{1}{8}}[\text{Sinc}(\mu)] \cdot 24\,\text{Sinc}(24v)$



5. (25 pts.)

Consider a spatial filter with point spread function $h[m,n]$ given below

$$
h[m,n] \qquad
\begin{array}{c|ccc}
 & & n & \\
 & -1 & 0 & 1 \\
\hline
-1 & -\frac{1}{3} & 1 & -\frac{1}{3} \\
m \quad 0 & -\frac{1}{3} & 1 & -\frac{1}{3} \\
1 & -\frac{1}{3} & 1 & -\frac{1}{3}
\end{array}
$$

a. (10) Find the output $g[m,n]$ when this filter is applied to the following input image

$$
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1
\end{array}
$$

b. (12) Find a simple expression for the magnitude $|H(\mu,\nu)|$ of the frequency response of this filter, and sketch it along the μ and ν axes, and the $\mu = \nu$ axis.

c. (3) Compare your results from parts a and b, and explain what this filter does. Relate spatial domain properties to frequency domain properties. There are three distinct, basic properties to discuss.

a.

$$
\begin{array}{c|cccccc}
 & \xrightarrow{n} & & & & & \\
m \downarrow & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
 & 0 & -\frac{2}{3} & \frac{4}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\
 & 0 & -1 & 2 & 1 & 1 & 1 \\
 & 0 & -1 & 2 & 1 & 1 & 1 \\
 & 0 & -1 & 2 & 1 & 1 & 1
\end{array}
$$

5. (continued)

b.

$$H(\mu,\nu) = \sum_{m=-1}^{1}\sum_{n=-1}^{1} h[m,n]\, e^{-j(2\pi\mu m + 2\pi\nu n)}$$

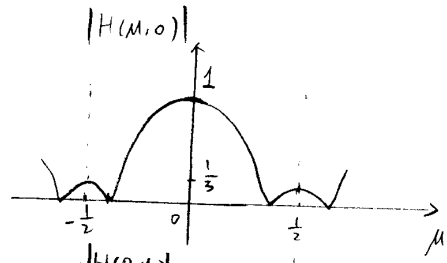
$$= \sum_{m=-1}^{1}\left(-\frac{1}{3}e^{-j2\pi(-1)\nu} + 1 + \left(-\frac{1}{3}\right)e^{-j2\pi(1)\nu}\right)e^{-j2\pi\mu m}$$

$$= \left(1 - \frac{1}{3}e^{j2\pi\nu} - \frac{1}{3}e^{-j2\pi\nu}\right)\cdot\left(e^{j2\pi\mu} + e^{-j2\pi\mu} + 1\right)$$

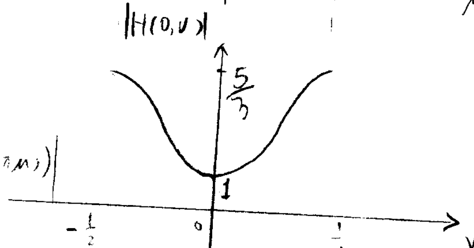
$$= \left(1 - \frac{2}{3}\cos(2\pi\nu)\right)\cdot\left(1 + 2\cos(2\pi\mu)\right)$$

$$|H(\mu,\nu)| = \left|1 - \frac{2}{3}\cos(2\pi\nu)\right| \cdot \left|1 + 2\cos(2\pi\mu)\right|$$

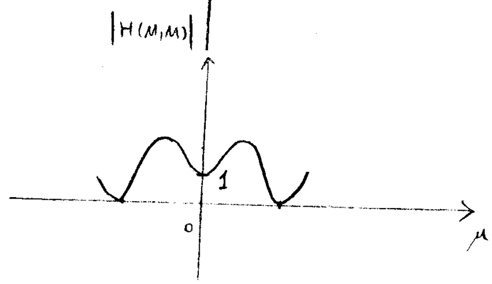
$$|H(\mu,0)| = \left|\frac{1}{3}\left(1 + 2\cos(2\pi\mu)\right)\right|$$



$$|H(0,\nu)| = 3\left|1 - \frac{2}{3}\cos(2\pi\nu)\right|$$
$$= 3 - 2\cos(2\pi\nu)$$



$$|H(\mu,\mu)| = \left|\left(1 - \frac{2}{3}\cos(2\pi\mu)\right)\cdot\left(1 + 2\cos(2\pi\mu)\right)\right|$$



c. $|H(\mu,0)|$ is a low pass filter

So $H(\mu,\nu)$ smoothes horizontal edges, e.g.: $\begin{matrix}0 & 0 & 0\\ 1 & 1 & 1\end{matrix} \rightarrow \begin{matrix}\frac{2}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{4}{3} & \frac{2}{3} & \frac{2}{3}\end{matrix}$

$|H(0,\nu)|$ enhances high frequencies, so $H(\mu,\nu)$ sharpens vertical edges.

e.g.: $\begin{matrix}0 & 1\\ 0 & 1\\ 0 & 1\end{matrix} \rightarrow \begin{matrix}-1 & 2\\ -1 & 2\\ -1 & 2\end{matrix}$

Because $\sum_m \sum_n h[m,n] = 1$ $H(\mu,\nu)$ has the DC preserving property, i.e., DC component of input $x[m,n]$ won't be changed by $h[m,n]$.