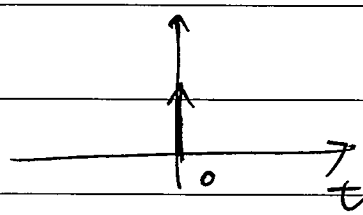
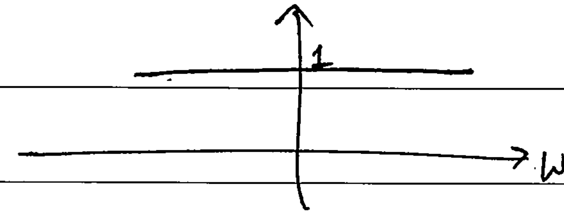
## Example 4.3

(time) $x(t) = \delta(t)$ $\xleftrightarrow{\mathcal{F}}$ (freq) $X(\omega) = 1 \quad , \forall \omega$





(proof) : $X(\omega) = \int_{-\infty}^{\infty} \underline{\delta(t)}\, e^{-j\omega t}\Big|_{t=0} dt = \int_{-\infty}^{\infty} \delta(t)\, \cancelto{1}{e^{0}}\, dt = 1$

- An impulse has equal energy at any freq.

  Intuition: narrow in time-domain

  $\rightarrow$ wide in freq-domain

- What about $x(t) = 1, \forall t \Rightarrow$ DC $\omega = 0$

  Conjecture $X(\omega) \propto \delta(\omega)$ since all energy is concentrated at $\omega = 0 \rightarrow$ no other freq content.

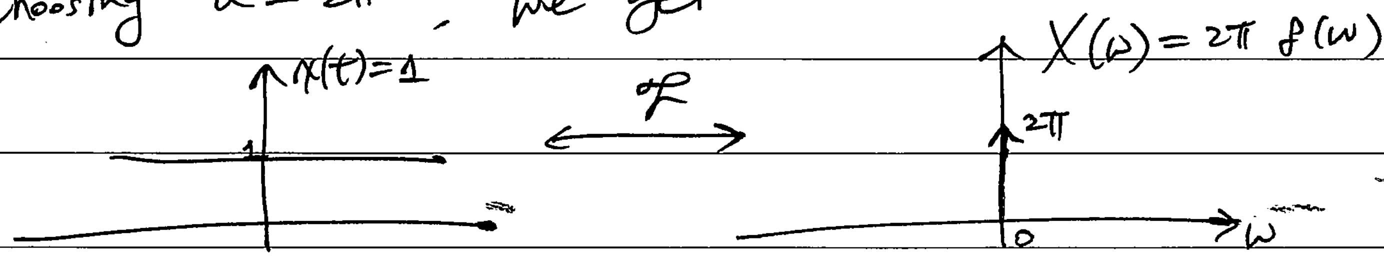
- validate by using Inverse FT

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

$$X(\omega) = \alpha\delta(\omega) \rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \alpha\,\delta(\omega)e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\underbrace{\int_{-\infty}^{\infty} \alpha\delta(\omega)e^{0}\,d\omega}_{=\alpha} = \frac{\alpha}{2\pi} \xrightarrow{\alpha = 2\pi} 1$$

Choosing $\alpha = 2\pi$, we get




" $x(t) = 1 \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = 2\pi\delta(\omega)$ "

\- Sinewaves

constant

$$x(t) = e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} X(\omega) = ?$$

- only freq. present is $\omega_0$

(all energy concentrated at $\omega_0$)

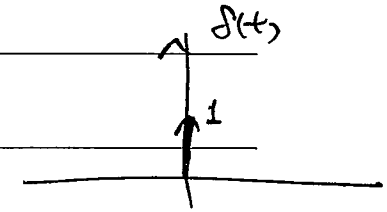
$\rightarrow$ conjecture $X(\omega) = 2\pi\,\underline{\delta(\omega-\omega_0)}$ shift.

- Check inverse FT :

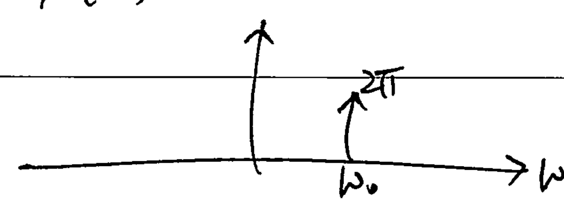
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(2\pi\,\delta(\omega-\omega_0)\right)e^{j\omega t}\,d\omega$$

$$= \frac{\cancel{2\pi}}{\cancel{2\pi}}\int_{-\infty}^{\infty}\delta(\omega-\omega_0)\,e^{j\omega_0 t}\,d\omega$$

$$= \Big\{\underbrace{\int_{-\infty}^{\infty}\delta(\omega-\omega_0)\,d\omega}_{=1}\Big\}\,e^{j\omega_0 t} = e^{j\omega_0 t}$$



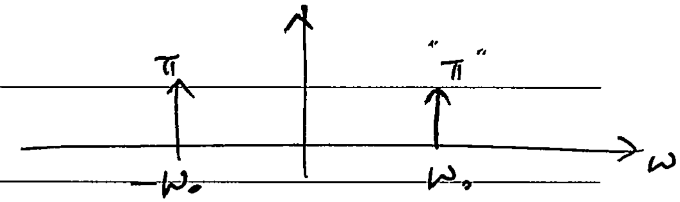

$$x(t) = e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} X(\omega) = 2\pi\,\delta(\omega-\omega_0)$$



Using Euler's formulas

+ the fact that FT is a linear operations.

$$\text{"}\cos(\omega_0 t) = \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t} \xleftrightarrow{\mathcal{F}} \pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)\text{"}$$




$$\text{"}\sin(\omega_0 t) = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t} \xleftrightarrow{\mathcal{F}} \frac{\pi}{j}\,\delta(\omega-\omega_0) - \frac{\pi}{j}\,\delta(\omega+\omega_0)\text{"}$$

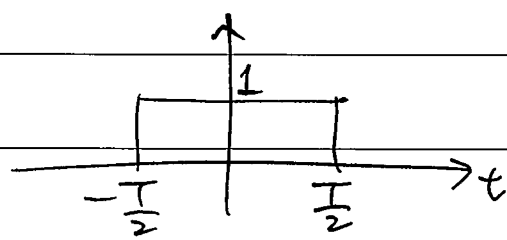
Example 4.4

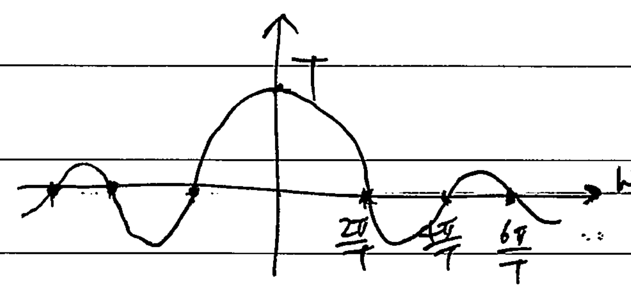
$$x(t) = \mathrm{rect}\left(\frac{t}{T}\right) \xleftrightarrow{\mathcal{F}} X(\omega) = \frac{\sin\left(T\frac{\omega}{2}\right)}{\frac{\omega}{2}}$$

$m\pi$

$\rightarrow 0$

$\rightarrow 0$



(proof)

- for value at $\omega = 0$, use L'Hospital's Rule :

$$X(\omega)\Big|_{\omega=0} = \frac{\frac{T}{2}\cos\left(T\frac{\omega}{2}\right)}{\frac{1}{2}}\Bigg|_{\omega=0} = T$$

- zero crossings at : $T\frac{\omega}{2} = m\pi$, for $m$ : integer

(because $\sin(m\pi) = 0$ )

- $X(\omega) \to 0$ as $\omega \to \infty$ or $-\infty$

(proof) $X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

$$= \int_{-\frac{T}{2}}^{\frac{T}{2}} 1 \cdot e^{-j\omega t}\, dt = \frac{-1}{j\omega} e^{-j\omega t}\Big|_{-T/2}^{T/2}$$

$$= \frac{-1}{j\omega}\left\{ e^{-j\omega\frac{T}{2}} - e^{j\omega\frac{T}{2}} \right\} = \frac{2}{\omega}\Big\{ \underbrace{\frac{1}{2j} e^{j\omega\frac{T}{2}} - \frac{1}{2j} e^{-j\omega\frac{T}{2}}}_{=\sin\left(T\frac{\omega}{2}\right)} \Big\}$$

$$= \frac{\sin\left(T\frac{\omega}{2}\right)}{\frac{\omega}{2}}$$

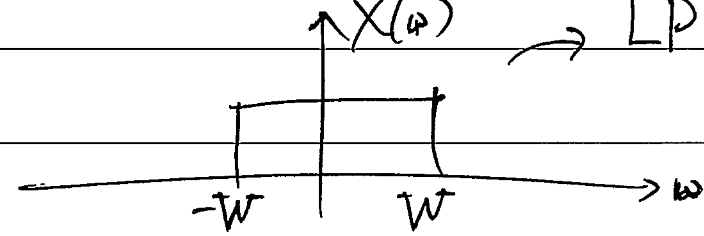
: purely real-valued because $x(t)$ is symmetric $(x(t) = x(-t))$

Example 4.5

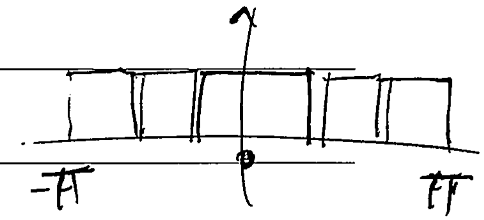
What if rectangle shape is in freq domain? (Low Pass Filter)

$\rightarrow$ LPF

$x(t) = ?$ $\longleftrightarrow$




Use Inverse FT




$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\, d\omega = \frac{1}{2\pi j t}\, e^{j\omega t}\Big|_{-W}^{W}$$

$$= \frac{1}{\pi t}\cdot\underbrace{\frac{1}{2j}\cdot\left\{e^{jWt} - e^{-jWt}\right\}}_{=\sin(Wt)} = \frac{\sin(Wt)}{\pi t}$$

$$\text{"}\ \frac{\sin(Wt)}{\pi t} \xleftrightarrow{\mathcal{F}} \mathrm{rect}\left(\frac{\omega}{2W}\right)\ \text{"}$$

## Symmetry Properties of FT

FT : $X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

Consider $x(t)$ : real-valued s.t. ( $x^*(t) = x(t)$ ) and take conjugate of both sides of eq.

$$X^*(\omega) = \int_{-\infty}^{\infty} x^*(t)\, e^{j\omega t}\, dt$$

Now, replace $\omega$ by $-\omega$ and invoke $x^*(t) = x(t)$

$$X^*(-\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = X(\omega)$$

$$\rightarrow \left( \begin{array}{l} X^*(-\omega) = X(\omega) \\ \text{or } X(-\omega) = X^*(\omega) \end{array} \right.$$

## Expressing in polar form

$$X(\omega) = |X(\omega)|\, e^{j\angle X(\omega)}$$

$$X^*(\omega) = |X(\omega)|\, e^{-j\angle X(\omega)}$$

We find that for $x(t)$: real-valued :

$$|X(-\omega)| = |X(\omega)| \Rightarrow \text{magnitude is even-symmetric}$$

$$\angle X(-\omega) = -\angle X(\omega) \Rightarrow \text{phase is odd-symmetric}$$

See Example 4.1 on pg 290-291

Fig 4.5 (a) $|X(\omega)|$ (b) $\angle X(\omega)$

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$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Holds for $a=-1$ $\quad x(-t) \overset{\mathcal{F}}{\longleftrightarrow} X(-\omega)$

Now consider $x(t)$: real & even-symmetric $\quad (x(-t) = x(t))$

Take FT at both sides : $X(-\omega) = X(\omega)$

Since $x(t)$ : real-valued : $X^*(\omega) = X(\omega)$ , $\forall \omega$

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Can only be true if $X(\omega)$ is real-valued

< $x(t)$ : real-valued & even-symmetric $\overset{\mathcal{F}}{\longleftrightarrow}$ $X(\omega)$ : real-valued & even-symmetric >

Consider $x(t)$ : real-valued & odd-symmetric

$$x(-t) = -x(t)$$

Take FT of both sides

$$X(-\omega) = -X(\omega)$$

Since $x(t)$ : real-valued.

$$X^*(\omega) = -X(\omega)$$

Can only be true if $X(\omega)$ is purely imaginary.

→ no real part

< $x(t)$ : real-valued & odd-symmetric $\overset{\mathcal{F}}{\longleftrightarrow}$ $X(\omega)$ : purely-imaginary & odd-symmetric >

∘ Note : Consider $x(t)$ : real & even s.t. $X(\omega)$ real & even

$$x(t-t_0) \underset{\mathcal{F}}{\longleftrightarrow} X(\omega)\underline{e^{-j\omega t_0}}$$

(about $t=0$)

If you shift in time so that $x(t-t_0)$ is not even-symmetric, the FT of $x(t-t_0)$ becomes complex-valued.