On the textbook

Prob 5.21 (d) $x[n] = 2^n \sin(\frac{\pi}{4}n)\, u[-n] \overset{DTFT}{\longleftrightarrow} X(\omega) = ?$

$$= 2^n u[-n] \left\{ \frac{1}{2j} e^{j\frac{\pi}{4}n} - \frac{1}{2j} e^{-j\frac{\pi}{4}n} \right\}$$

$$= \underline{\left(\frac{1}{2}\right)^{-n} u[-n] \frac{1}{2j}} \left\{ e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n} \right\}$$

prop. : ∘ $a^n u[n] \underset{DTFT}{\longleftrightarrow} \frac{1}{1 - ae^{-j\omega}}$ ∘ $x[-n] \overset{DTFT}{\longleftrightarrow} X(-\omega)$

∘ $e^{j\omega_0 n} x[n] \underset{DTFT}{\longleftrightarrow} X(\omega - \omega_0)$

"$\frac{\pi}{4}$"

$$X(\omega) = \frac{1}{2j}\left\{ \frac{1}{1 - \frac{1}{2}e^{-j(-(\omega - \frac{\pi}{4}))}} - \frac{1}{1 - \frac{1}{2}e^{-j(-(\omega + \frac{\pi}{4}))}} \right\}$$

$$= \frac{1}{2j}\left\{ \frac{1}{1 - \frac{1}{2}e^{j(\omega - \frac{\pi}{4})}} - \frac{1}{1 - \frac{1}{2}e^{j(\omega + \frac{\pi}{4})}} \right\}$$

5.21 (e) $x[n] = \left(\frac{1}{2}\right)^{|n|} \cos\left(\frac{\pi}{8}(n-1)\right)$

prop. : $a^{|n|} \overset{DTFT}{\longleftrightarrow} \dfrac{1-a^2}{1-2a\cos w + a^2}$

$$x[n] = \left(\frac{1}{2}\right)^{|n|} \frac{1}{2} e^{j\frac{\pi}{8}n} e^{-j\frac{\pi}{8}} + \left(\frac{1}{2}\right)^{|n|} \frac{1}{2} e^{j\frac{\pi}{8}} e^{-j\frac{\pi}{8}n}$$

Thus,

$$X(w) = \frac{1}{2} \frac{1-\left(\frac{1}{2}\right)^2}{1-2\left(\frac{1}{2}\right)\cos\left(w-\frac{\pi}{8}\right)+\left(\frac{1}{2}\right)^2} e^{-j\frac{\pi}{8}}$$

$$+ \frac{1}{2} \frac{1-\left(\frac{1}{2}\right)^2}{1-2\left(\frac{1}{2}\right)\cos\left(w+\frac{\pi}{8}\right)+\left(\frac{1}{2}\right)^2} e^{j\frac{\pi}{8}}$$

5.21 (f) $x[n] = \begin{cases} n, & -3 \le n \le 3 \\ 0, & o/w \end{cases}$

$x[n] = n\,\tilde{x}[n]$ where $\tilde{x}[n] = u[n+3] - u[n-4]$

The DTFT of $\tilde{x}[n]$ is $\tilde{X}(w) = \dfrac{\sin\left(\frac{7}{2}w\right)}{\sin\left(\frac{1}{2}w\right)}$

prop. : $n\,x[n] \overset{DTFT}{\longleftrightarrow} j\,\dfrac{dX(\omega)}{d\omega}$

$(uv)' = uv' + u'v$

Thus, $X(\omega) = j\,\dfrac{d}{d\omega}\left\{ \dfrac{\sin(\frac{n}{2}\omega)}{\sin(\frac{1}{2}\omega)} \right\}$

$= j\,\dfrac{1}{\sin^2(\frac{\omega}{2})}\left\{ \frac{n}{2}\cos(\frac{n}{2}\omega)\sin(\frac{1}{2}\omega) - \sin(\frac{n}{2}\omega)\frac{1}{2}\cos(\frac{1}{2}\omega) \right\}$

5.21 (g) $\quad x[n] = \sin(\frac{\pi}{2}n) + \cos(n)$

$\cancel{\sum \delta(\omega + \frac{\pi}{2} + 2\pi k)}$

$X(\omega) = \dfrac{2\pi}{2j}\left\{\delta(\omega - \frac{\pi}{2}) - \delta(\omega + \frac{\pi}{2})\right\} + \dfrac{2\pi}{2}\left\{\delta(\omega-1) + \delta(\omega+1)\right\}$

for $-\pi < \omega < \pi$

(repeats every 2π)

5.21 (h) $\quad x[n] = \sin(\frac{5\pi}{3}n) + \cos(\frac{7\pi}{3}n)$

$= \sin\left(\left(\frac{5\pi}{3} - \frac{6\pi}{3}\right)n\right) + \cos\left(\left(\frac{7\pi}{3} - \frac{6\pi}{3}\right)n\right)$

$= -\sin(\frac{\pi}{3}n) + \cos(\frac{\pi}{3}n)$

$X(\omega) = -\dfrac{2\pi}{2j}\left(+\delta(\omega - \frac{\pi}{3}) - \delta(\omega + \frac{\pi}{3})\right) + \dfrac{2\pi}{2}\left(\delta(\omega - \frac{\pi}{3}) + \delta(\omega + \frac{\pi}{3})\right)$

for $-\pi < \omega < \pi$

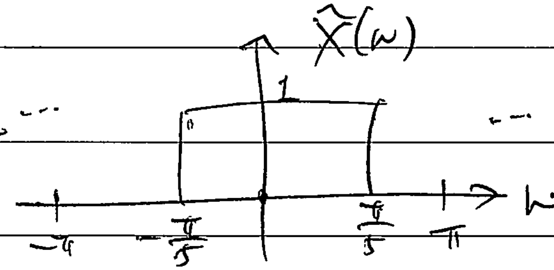
(repeats every 2π)

5.21 (k) $x[n] = \dfrac{\sin\left(\frac{\pi}{5}n\right)}{\pi n} \cos\left(\frac{7\pi}{2} n\right)$

$$\cos\left(\frac{7\pi}{2}n\right) = \cos\left(\left(\frac{7\pi}{2} - \frac{8\pi}{2}\right)n\right) = \cos\left(-\frac{\pi}{2}n\right) = \cos\left(\frac{\pi}{2}n\right)$$

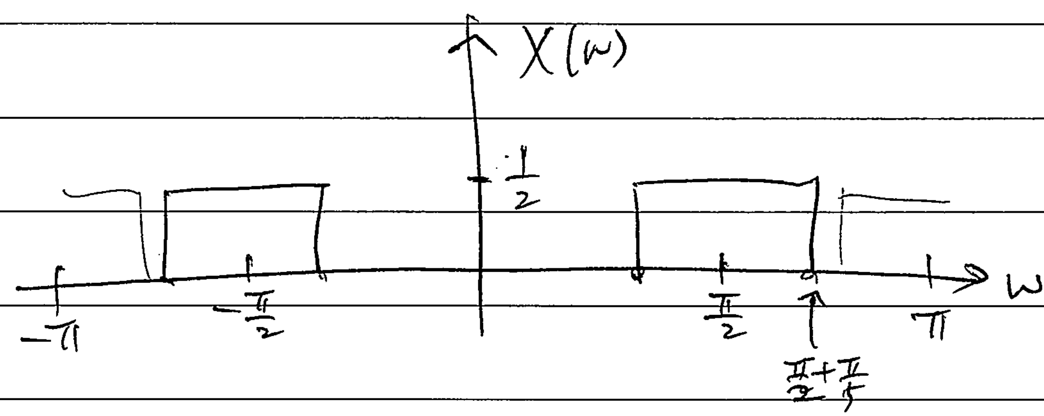
$$X(\omega) = \frac{1}{2}\tilde{X}\left(\omega - \frac{\pi}{2}\right) + \frac{1}{2}\tilde{X}\left(\omega + \frac{\pi}{2}\right)$$

$\dfrac{\sin\left(\frac{\pi}{5}n\right)}{\pi n} \overset{\mathcal{F}}{\longleftrightarrow}$



$* \left(\delta\left(\omega - \frac{\pi}{2}\right) + \delta\left(\omega + \frac{\pi}{2}\right)\right)$

Thus,



Prob 5.35 **All-Pass Filter**

(Digital Filter)

$y[n] - a\,y[n-1] = b\,x[n] + x[n-1]$: difference eq.

Freq. Response $H(\omega) = ?$

- Take DTFT of both sides:

$$Y(\omega)\left(1 - a e^{-j\omega}\right) = X(\omega)\left(b + e^{-j\omega}\right)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{b + e^{-j\omega}}{1 - a e^{-j\omega}}$$

convolution prop.

$$\begin{pmatrix} Y = HX \\ y = h * x \end{pmatrix}$$

- Consider $b = -a$, where a : real-valued.

$$H(\omega) = \frac{-a + e^{-j\omega}}{1 - a e^{-j\omega}} = \frac{e^{-j\omega}\left(1 - a e^{j\omega}\right)}{1 - a e^{-j\omega}}$$

- since a is real-valued,

$$\frac{1 - a e^{j\omega}}{1 - a e^{-j\omega}} = \frac{c}{c^*}$$, for any complex number c,

$\frac{c}{c^*}$ has magnitude 1.

So it's easy to see in polar form

$$\frac{c}{c^*} = \frac{\cancel{|c|}\, e^{j\angle c}}{\cancel{|c|}\, e^{-j\angle c}} = e^{j2\angle c}$$

$$|H(\omega)| = \underbrace{|e^{-j\omega}|}_{=1}\ \underbrace{\left|\frac{1-ae^{j\omega}}{1-ae^{-j\omega}}\right|}_{=1}$$

, since $|ab| = |a||b|$

$= 1$ for all ω

Thus: $y[n] - ay[n-1] = -ax[n] + x[n-1]$ is an all-pass (magnitude) filter for any value of a (real-valued)

$|H(\omega)| = 1, \ \forall \omega$

For any sinewave into this system, amplitude is unchanged.
(only phase changes)

$$Ae^{j(\omega_0 n + \theta)} \longrightarrow \boxed{\text{All-pass Filter}} \longrightarrow Ae^{j(\omega_0 n + \theta + \phi(\omega_0))}$$

$\phi(\omega_0)$: $\angle H(\omega_0)$