4.9 Consider the random signal given by $A(t) = D(t - U)$, $-\infty < t < \infty$, where U is a random variable that is uniformly distributed on the interval $[0, T]$ and the random signal $D(t)$ is given by (4.24). Suppose that the random sequence (A_n) is wide-sense stationary, is independent of U, and has autocorrelation function $\rho_k = E\{A_0 A_k\}$, $-\infty < k < \infty$.

(a) Show that $A(t)$ is wide-sense stationary and has autocorrelation function given by

$$R_A(\tau) = \sum_{k=-\infty}^{\infty} \rho_k \, r_\zeta(\tau - k\,T),$$

where

$$r_\zeta(\tau) = T^{-1} \int_{-\infty}^{\infty} \zeta(t)\, \zeta(t+\tau)\, \mathrm{d}t.$$

(b) Apply the result obtained in part **(a)** to find the autocorrelation and spectral density functions for the random signal $A(t)$ if $\zeta(t) = p_T(t)$, $\rho_0 = \alpha^2$, $\rho_1 = \rho_{-1} = \beta$, and $\rho_k = 0$ for $|k| \geq 2$.

(c) For $\alpha = 1$ and $\beta = 1/2$, compare a plot of the spectral density obtained in part **(b)** with the spectral density that is obtained if the sequence (A_n) is a sequence of independent zero-mean random variables with $E\{A_n^2\} = 1$.

MBP 4.9

$A(t) \triangleq D(t-u)$ u is unif. rv on $[0,T]$

$D(t) = \sum_{n=-\infty}^{\infty} A_n \xi(t-nT)$ $\xi(t)$ is a deterministic, finite energy pulse

$\{A_n\}$ is a WSS discrete-time random proc. with auto correlation

$$\rho_k = E\{A_k A_0\} \qquad k \in \mathbb{Z}$$

Then rvs $\{A_n\}$ are statistically indep of U.

(a) Show $A(t)$ is WSS with

$$R_A(\tau) = \sum_{k=-\infty}^{\infty} \rho_k \, r_\xi(\tau - kT) \; ; \qquad r_\xi(\tau) = \frac{1}{T}\int_{-\infty}^{\infty} \xi(t+\tau)\xi(t)\,dt$$

Let $\mu \triangleq E\{A_n\}$ a const. indep. on n since seq. is WSS.

$$E\{A(t)\} = \sum_n E\{A_n \xi(t-U-nT)\} = \sum_n E\{A_n\} E\{\xi(t-U-nT)\}$$

$$= \mu \sum_{n=-\infty}^{\infty} E\{\xi(t-u-nT)\}$$

$$= \mu \sum_{n=-\infty}^{\infty} \frac{1}{T}\int_0^T \xi(t-u-nT)\,du \quad \leftarrow \text{c.o.v. } v = u+nT$$

$$= \mu \sum_{n=-\infty}^{\infty} \frac{1}{T}\int_{nT}^{(n+1)T} \xi(t-v)\,dv = \frac{\mu}{T}\int_{-\infty}^{\infty} \xi(t-v)\,dv$$

$$= \frac{\mu}{T}\int_{-\infty}^{\infty} \xi(s)\,ds \quad \text{which does not dep. on } t.$$

$$E\{A(t+\tau)A(t)\} = E\Big\{\sum_n A_n \xi(t+\tau-U-nT) \sum_k A_k \xi(t-U-kT)\Big\}$$

$$= \sum_n \sum_k \underbrace{E\{A_n A_k\}}_{\rho_{n-k}} E\{\xi(t+\tau-U-nT)\xi(t-U-kT)\}$$

In the double sum make the substitution $j = n-k$

$$= \sum_n \sum_j \rho_j \underbrace{E\{\xi(t+\tau-U-nT)\xi(t-U-nT+jT)\}}$$

$$= \frac{1}{T}\int_0^T \xi(t+\tau-u-nT)\xi(t-u-nT+jT)\,du$$

c.o.v.
$v = u+nT$

$$= \frac{1}{T}\int_{nT}^{(n+1)T} \xi(t+\tau-v)\xi(t-v+jT)\,dv$$

$\therefore$

$$R_A(t+\tau,t) = \sum_n \sum_j \rho_j \frac{1}{T}\int_{nT}^{(n+1)T} \xi(t+\tau-v)\xi(t-v+jT)\,dv$$

$$= \frac{1}{T}\sum_j \rho_j \sum_n \int_{nT}^{(n+1)T} \cdots \text{above} \cdots$$

$$= \frac{1}{T}\sum_j \rho_j \int_{-\infty}^{\infty} \xi(t+\tau-v)\xi(t-v+jT)\,dv$$

c.o.v.
$x = t-v+jT$

$$= \frac{1}{T}\sum_j \rho_j \underbrace{\int_{-\infty}^{\infty} \xi(x+\tau-jT)\xi(x)\,dx}_{\Gamma_\xi(\tau-jT)}$$

35500

$\therefore$ $R_A(t+\tau, t)$ does not really depend on t. Hence write

$$R_A(\tau) = \frac{1}{T} \sum_{j=-\infty}^{\infty} \rho_j \, r_\xi(\tau - jT). \qquad (*)$$

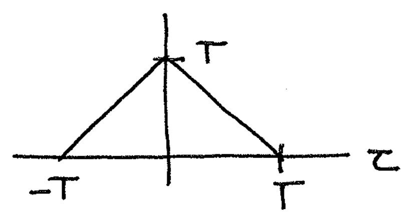
(b) Find $R_A(\tau) \longleftrightarrow S_A(f)$ for case where

$\xi(t) = p_T(t)$

$\rho_0 = \alpha^2$

$\rho_1 = \rho_{-1} = \beta$

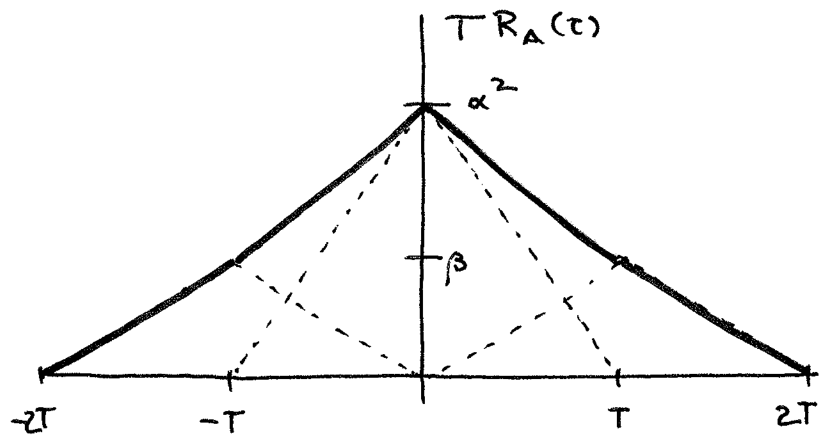
$\rho_k = 0$ for $|k| \geq 2$.

$r_\xi(\tau) = \tilde{p}_T * p_T(\tau) =$ 

Eq. (*) reduced to

$$R_A(\tau) = \frac{1}{T} \beta \, r_\xi(\tau+T) + \frac{1}{T} \alpha^2 r_\xi(\tau) + \frac{1}{T} \beta \, r_\xi(\tau - T)$$

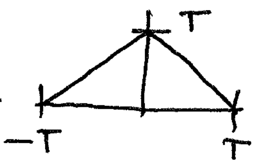
Plotted this autocorrelation looks like



The power spectral density is the Fourier Transform of the auto-correlation

$$S_A(f) \longleftrightarrow R_A(\tau)$$

Start by finding F.T. of triangular pulse and then use linearity and time shift properties of F.T.

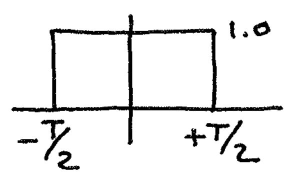


The F.T. is

$$\int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt = X(f) \longleftrightarrow x(t) = \int_{-\infty}^{\infty} X(f)\, e^{+j2\pi f t}\, df$$

Start with rec. pulse

$$x(t) =$$



$$\longleftrightarrow X(f) = \int_{-T/2}^{T/2} e^{-j2\pi f t}\, dt$$

$$= -\frac{1}{j2\pi f} e^{-j2\pi f t}\Big|_{t=-T/2}^{T/2} = \frac{e^{j\pi f T} - e^{-j\pi f T}}{j2\pi f}$$

$$= T\,\frac{\sin(\pi f T)}{\pi f T} \triangleq T\,\mathrm{sinc}(fT)$$

Then note that $r_\xi(t) = \tilde{x} * x(t) = x * x(t)$

$$\Longrightarrow R_\xi(f) = X^2(f)$$

$$= T^2\,\mathrm{sinc}^2(fT)$$

Now using linearity and time shift properties of FT

$$S_A(f) = \frac{1}{T}\beta\, e^{+j2\pi f T} R_\xi(f) + \frac{1}{T}\alpha^2 R_\xi(f) + \frac{1}{T}\beta\, e^{-j2\pi f T} R_\xi(f)$$

$$S_A(f) = \frac{1}{T} R_\xi(f) \left[\beta \underbrace{\left(e^{+j2\pi fT} + e^{-j2\pi fT} \right)}_{2\cos 2\pi fT} + \alpha^2 \right]$$

$$= \left[\alpha^2 + 2\beta \cos 2\pi fT \right] T \,\mathrm{sinc}^2(fT)$$

(c) If $\alpha = 1$, $\beta = 1/2$ above expression is

$$S_A(f) = \left[1 + \cos(2\pi fT) \right] T \,\mathrm{sinc}^2(fT) \qquad (1)$$

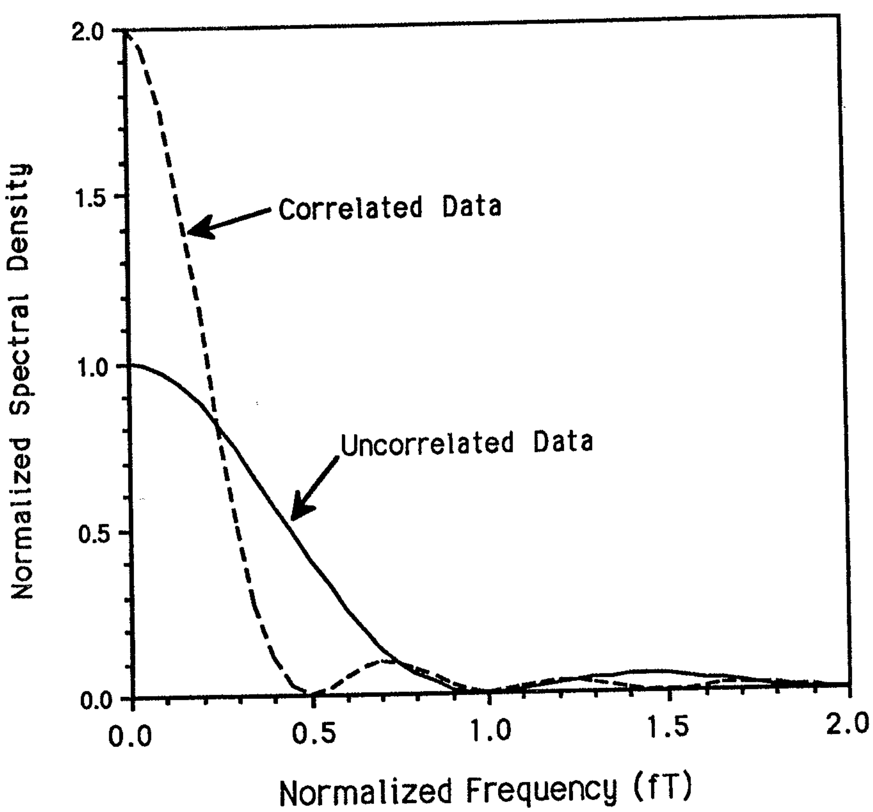
$$= 2T\cos^2(\pi fT)\,\mathrm{sinc}^2(fT) = 2T\,\mathrm{sinc}^2(2fT)$$

On the otherhand if symbol seq. A_n is uncorrelated zero mean with $E(A_n^2) = 1 \implies \rho_0 = 1,\ \rho_k = 0 \quad k \neq 0$

$$\implies R_A(\tau) = \frac{1}{T} \rho_0 r_\xi(\tau)$$

$$\updownarrow$$

$$S_A(f) = T\,\mathrm{sinc}^2(fT) \qquad (2)$$



Spectral Densities for Correlated and Uncorrelated Data Signals

5.3 Consider the binary communication system of Figure 5-2. (See Section 5.2.1 for a description of this system.) The signals $s_0(t)$ and $s_1(t)$ are defined by

$$s_0(t) = 4\, p_T(t)$$

and

$$s_1(t) = 2\, p_T(t).$$

Suppose $T = 2$ and $N_0 = 4$.

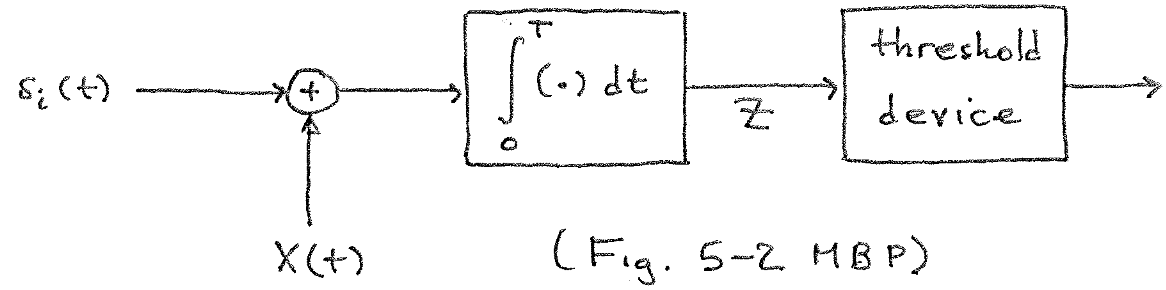
(a) Find the error probabilities $P_{e,0}$ and $P_{e,1}$ for the system with threshold $\gamma = 7$.

(b) What is the minimax threshold for this system?

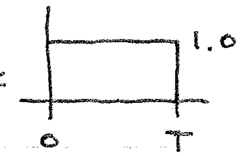
(c) Find the minimax error probability.

(d) Add a squaring device to the system at the input to the threshold device. The decision statistic is now Z^2 rather than Z. Find expressions in terms of the threshold γ and the function Q for the error probabilities $P_{e,0}$ and $P_{e,1}$ for this new system. (*Warning*: Z^2 is *not* Gaussian.)

MBP 5.3



(Fig. 5-2 MBP)

$s_0(t) = 4\, p_T(t)$; $s_1(t) = 2\, p_T(t)$; $p_T(t) =$ 

$T = 2$, $N_0 = 4$ (presumably $X(t)$ is AWGN with $R_X(\tau) = \frac{N_0}{2}\delta(\tau)$)

(a) Find $P_{e,0}$ and $P_{e,1}$ for $\gamma = 7$.

$$Z = \underbrace{\int_0^T s_i(t)\, dt}_{} + \int_0^T X(t)\, dt$$

$$\int_0^T s_0(t)\, dt = 4\int_0^T p_T(t)\, dt = 4T = 8$$

$$\int_0^T s_1(t)\, dt = 2\int_0^T p_T(t)\, dt = 4$$

This is a r.v. with 0 mean and variance

$$\int_0^T\int_0^T E\{X(t)X(s)\}\, dt\, ds = \frac{N_0}{2}\int_0^T\int_0^T \delta(t-s)\, ds\, dt$$

$$= \frac{N_0}{2}\int_0^T dt = \frac{N_0}{2}\cdot T = \frac{4}{2}\cdot 2 = 4$$

It is Gaussian $\sim N(0, 4)$.

If s_0 Tx $\Longrightarrow$ $Z \sim N(8,4)$
" s_1 Tx $\Longrightarrow$ $Z \sim N(4,4)$.

For the dec. rule with $\gamma = 7$

$$Z \underset{\leq}{>} 7 \quad \begin{array}{l} \text{decide } s_0 \\ \quad " \quad\quad s_1 \end{array}$$

$$P_{e,0} = P(Z \leq 7 \mid s_0)$$
$$= P\left(\frac{Z-8}{2} \leq \frac{7-8}{2} \mid s_0\right)$$
$$= \Phi\left(-\tfrac{1}{2}\right) = Q\left(\tfrac{1}{2}\right) \approx 0.309$$

$$P_{e,1} = P(Z > 7 \mid s_1)$$
$$= P\left(\frac{Z-4}{2} > \frac{7-4}{2} \mid s_1\right)$$
$$= Q\left(\tfrac{3}{2}\right) \approx 0.067$$

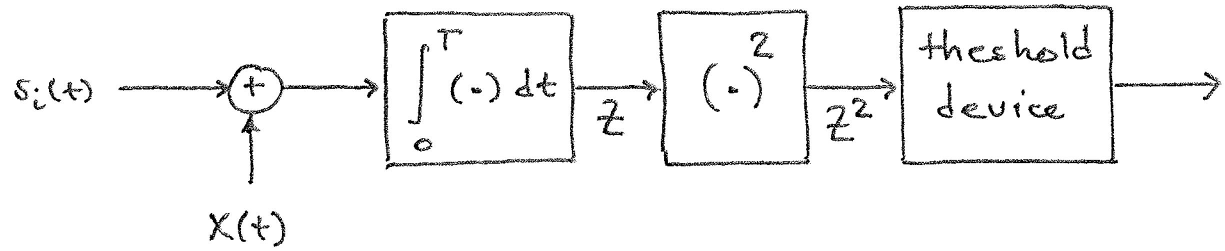
(b) What is minimax threshold for this prob?

$$\gamma_m = \frac{\mu_0 + \mu_1}{2} = \frac{8+4}{2} = 6.$$

(c) Find $P^*_{e,m}$

$$P^*_{e,m} = Q\left(\frac{\mu_0 - \gamma_m}{\sigma}\right) = Q\left(\frac{8-6}{2}\right) = Q(1)$$
$$\approx 0.159$$

(d) Change system to



Find $P_{e,0}$ and $P_{e,1}$ as functs. of γ.

The two poss. dists. of Z remain unchanged. The test is now of form

$$Z^2 \begin{array}{c} > \\ \le \end{array} \gamma \quad \begin{array}{l} \text{decide } s_0 \\ \quad '' \quad\ s_1 \end{array}$$

(of course Z^2 is not Gaussian).

Only makes sense to consider thresholds $\gamma \geq 0$.

$$P_{e,0}(\gamma) = P\left(Z^2 \le \gamma \mid s_0\right)$$

$$= P\left(-\sqrt{\gamma} < Z \le \sqrt{\gamma} \mid s_0\right) \longrightarrow$$ reduce back to consideration of Gauss. dist.

$$= P\left(\frac{-\sqrt{\gamma}-8}{2} < \frac{Z-8}{2} \le \frac{\sqrt{\gamma}-8}{2} \,\Big|\, s_0\right)$$

$$= \Phi\left(\frac{\sqrt{\gamma}-8}{2}\right) - \Phi\left(\frac{-\sqrt{\gamma}-8}{2}\right).$$

$$= Q\left(4 - \frac{\sqrt{\gamma}}{2}\right) - Q\left(4 + \frac{\sqrt{\gamma}}{2}\right).$$

$$P_{e,1}(\gamma) = P\left(Z^2 > \gamma \mid s_1\right)$$

$$= P\left(Z > \sqrt{\gamma} \mid s_1\right) + P\left(Z < -\sqrt{\gamma} \mid s_1\right)$$

$$= P\left(\frac{Z-4}{2} > \frac{\sqrt{\gamma}-4}{2} \mid s_1\right) + P\left(\frac{Z-4}{2} < \frac{-\sqrt{\gamma}-4}{2} \mid s_1\right)$$

$$= Q\left(\frac{\sqrt{\gamma}-4}{2}\right) + \Phi\left(\frac{-\sqrt{\gamma}-4}{2}\right)$$

$$= Q\left(\frac{\sqrt{\gamma}}{2}-2\right) + Q\left(\frac{\sqrt{\gamma}}{2}+2\right).$$

5.6 The received signal in a binary baseband data communications system is given by $Y(t) = s_i(t) + X(t)$, where $X(t)$ is a stationary Gaussian random process with mean 1 and autocovariance function given by

$$C_X(u) = 4\exp(-|u|).$$

The signals $s_i(t)$ are such that $s_0(t) = 3$ for $0 < t < T$ and $s_1(t) = 1$ for $0 < t < T$; for t not in the interval $(0, T)$, $s_0(t) = s_1(t) = 0$. The receiver samples the received signal at times $T/4$, $T/2$, and $3T/4$ and sums the sample values; that is, it forms the statistic Z, which is the random variable given by

$$Z = Y(T/4) + Y(T/2) + Y(3T/4).$$

(a) First suppose the threshold is 6. That is, if $Z > 6$, the receiver decides that 0 was sent; however, if $Z < 6$, it decides that 1 was sent. Find the probability of error when 0 is transmitted and the probability of error when 1 is transmitted. Express these two probabilities in terms of the function Q. For one of these, you should be able to obtain a numerical value without the aid of a table or calculator.

(b) Is 6 the optimum threshold in the minimax sense? If not, what is the optimum minimax threshold? (Be careful—the noise has a nonzero mean.)

(c) Check your answer to part **(b)** by comparing the two error probabilities. For your value of the threshold, is the probability of error when 0 is sent equal to the probability of error when 1 is sent? Should these two probabilities be equal? Explain why or why not.

MBP Problem 5.6

$Y(t) = s_i(t) + X(t)$

$X(t)$: Gaussian

$EX(t) = 1$

$C_X(\tau) = 4e^{-|\tau|}$

$s_0(t) = 3\,p_T(t)$

$s_1(t) = p_T(t)$

Receiver samples $Y(t)$ to form decision statistic

$$Z = Y(T/4) + Y(T/2) + Y(3T/4)$$

(a) Suppose the test is

$$Z \underset{<}{\overset{>}{}} 6 \quad \begin{array}{l} \text{decide } s_0 \\ \text{decide } s_1 \end{array}$$

Find $P_{e,0}$ and $P_{e,1}$.

First must find distribution of Z under the two hypotheses.

Under H_0

Signal part of Z is $s_0(T/4) + s_0(T/2) + s_0(3T/4)$
$= 9$

Under H_1

Signal part of Z is $s_1(T/4) + s_1(T/2) + s_1(3T/4)$
$= 3$

Noise part of Z is same under H_0 and H_1. It is

$$X(T/4) + X(T/2) + X(3T/4) \triangleq \tilde{X}$$

$\tilde{X}$ is Gaussian.

$$E\tilde{X} = 3$$

$$E\tilde{X}^2 = E\left\{\left(X(T/4) + X(T/2) + X(3T/4)\right)^2\right\}$$

$$= E\{X^2(T/4)\} + E\{X^2(T/2)\} + E\{X^2(3T/4)\}$$
$$+ 2E\{X(T/4)X(T/2)\} + 2E\{X(T/4)X(3T/4)\}$$
$$+ 2E\{X(T/2)X(3T/4)\}$$

$$= 3R_X(0) + 2R_X(T/4) + 2R_X(T/2) + 2R_X(T/4)$$

$$= 3R_X(0) + 4R_X(T/4) + 2R_X(T/2)$$

$$C_X(\tau) = R_X(\tau) - (EX(t))^2 \Longrightarrow R_X(\tau) = 1 + C_X(\tau)$$
$$= R_X(\tau) - 1$$

$$R_X(0) = 1 + C_X(0) = 5$$
$$R_X(T/4) = 1 + 4e^{-T/4}$$
$$R_X(T/2) = 1 + 4e^{-T/2}$$

$$Var(\tilde{X}) = E(\tilde{X}^2) - (E\tilde{X})^2$$
$$= 15 + 4 + 16e^{-T/4} + 2 + 8e^{-T/2} - 9$$
$$= 12 + 16e^{-T/4} + 8e^{-T/2}$$

Under H_0

$$Z = 9 + \tilde{X} \sim N\left(12, 12 + 16e^{-T/4} + 8e^{-T/2}\right)$$

Under H_1

$$Z = 3 + \tilde{X} \sim N\left(6, 12 + 16e^{-T/4} + 8e^{-T/2}\right).$$

$$P_{e,0} = P(Z < 6 \mid H_0)$$

$$= P\left(\frac{Z-12}{\sigma} < \frac{6-12}{\sigma} \,\middle|\, H_0\right) \qquad \sigma = 2\sqrt{3 + 4e^{-T/4} + 2e^{-T/2}}$$

$$= \Phi(-6/\sigma) = Q(6/\sigma)$$

$$P_{e,1} = P(Z > 6 \mid H_1)$$

$$= P\left(\frac{Z-6}{\sigma} > \frac{6-6}{\sigma} \,\middle|\, H_1\right)$$

$$= Q(0) = 1/2$$

(b) No because $P_{e,0} \neq P_{e,1}$ for threshold equal to 6. If the noise had zero mean, 6 would have been minimax threshold.

To find minimax threshold we need to find average of means of Z under the two hypotheses

$$\gamma_m = \frac{E(Z \mid H_0) + E(Z \mid H_1)}{2} = \frac{12+6}{2} = 9$$

(c) with $\gamma = 9$

$$P_{e,0} = P\left(\frac{Z-12}{\sigma} < \frac{9-12}{\sigma} \,\middle|\, H_0\right) = Q(3/\sigma)$$

$$P_{e,1} = P\left(\frac{Z-6}{\sigma} > \frac{9-6}{\sigma} \,\middle|\, H_1\right) = Q(3/\sigma)$$

$\Longrightarrow$ $P_{e,0} = P_{e,1}$ as the must for the minimax soln.

A.5. Assuming a bandwidth of 2 MHz, find the rms noise voltage across the output terminals of the circuit shown in Figure A.9 if it is at a temperature of 400 K.

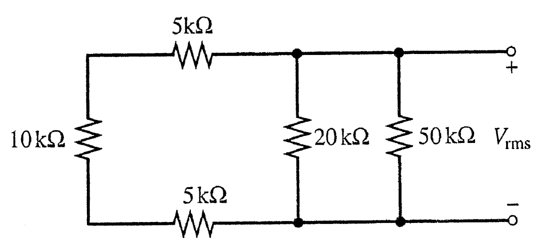
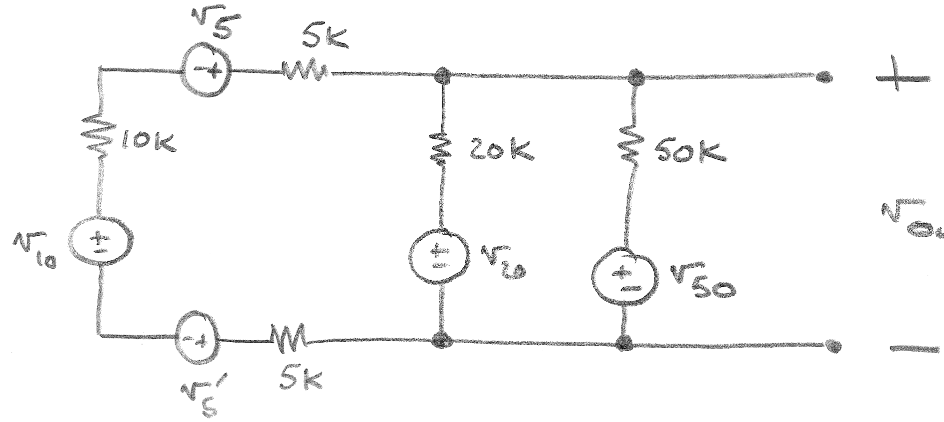


Figure A.9

Z+T Problem A.5

There are two ways to attack this and they should lead to the same answer.

The Long Way Model each noisy resistor as an indep noise source and solve



T = 400K. The five noise sources are statistically independent, zero mean, white random processes. Their two-sided power spectral densities are

$$S_{v_5}(f) = 2kTR_5 = S_{v_5'}(f) \quad \forall f \qquad R_5 = 5k\Omega$$

$$S_{v_{10}}(f) = 2kTR_{10} \quad \forall f \qquad R_{10} = 10k\Omega$$

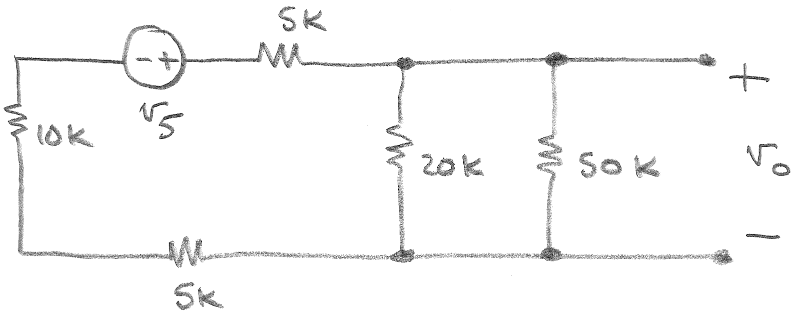
$$S_{v_{20}}(f) = 2kTR_{20} \quad \text{"} \qquad R_{20} = 20k\Omega$$

$$S_{v_{50}}(f) = 2kTR_{50} \quad \text{"} \qquad R_{50} = 50k\Omega$$

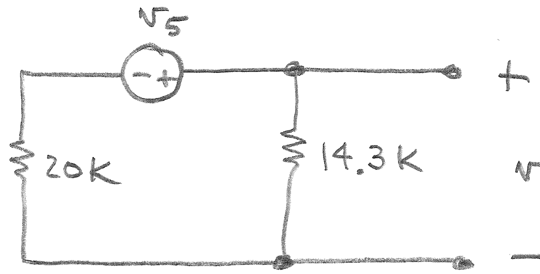
Superposition of voltages always works for LTI systems but what we want is superposition of powers and this also works here because of the statistical independance of the individual noise sources.

To use this method we need the transfer function from each input to the output. These we get from the usual 201 method.

T.F. from 5 $\rightarrow$ output: $H_{5,out}$



Simply using circuit theory



$$v_{out} = \underbrace{\frac{14.3}{20 + 14.3}}_{H_{5,out} = 0.42} v_5$$

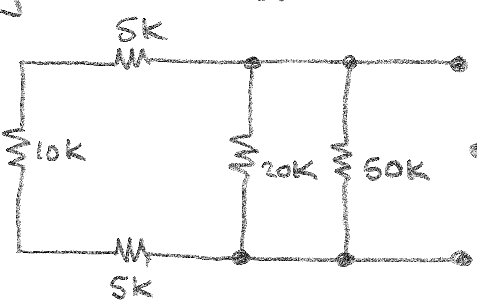
Then if find all the others

$$S_{v_{out}}(f) = H^2_{5,out} S_{v_5}(f) + H^2_{5',out} S_{v_5'}(f)$$
$$+ H^2_{10,out} S_{v_{10}}(f) + H^2_{20,out} S_{v_{20}}(f) + H^2_{50,out} S_{v_{50}}(f)$$

Then the power in a 2MHz one-sided BW would be

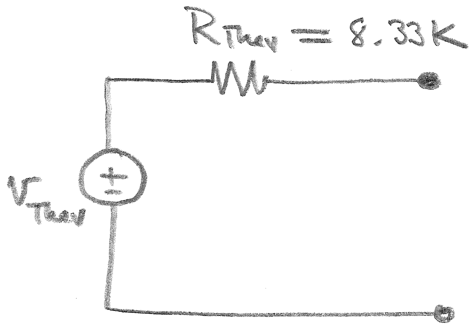
$$P_{2M} = 2 \int_{\substack{2M \\ BW}} S_{v_{out}}(f)\, df = 2 \cdot (2MHz) \cdot S_{v_{out}, height}$$

The Short Way Use Nyquist's Theorem to find a single resistor



$\leftarrow R_{th} = 20K \,||\, 20K \,||\, 50K$
$= 10K \,||\, 50K$
$= 8.33K$

Then this noisy resistor produces the same noise voltage as the previous



$$S_{V_{Thev}}(f) = 2kTR_{Thev} \quad \forall f$$

$$[V^2/Hz]$$

Then over a 2MHz one-sided BW

$$P = 2(2\times10^6)(2)(1.38\times10^{-23})(400)(8.33\times10^3)$$

$$= (3200\times10^6)(1.38\times10^{-23})(8.33\times10^3)$$

$$\begin{array}{r} -23 \\ +12 \\ \hline -11 \end{array}$$

$$= (3.2)(1.38)(8.33)\times10^{-11} \quad V^2$$

$$= 36.79\times10^{-11}\ V^2$$

$\Longrightarrow$ RMS noise voltage $= \sqrt{P} = 19.2\ \mu V$ rms.

A.7. A source with equivalent noise temperature $T_s = 1000$ K is followed by a cascade of three amplifiers having the specifications shown in Table A.1. Assume a bandwidth of 50 kHz.

Table A.1

Amplifier no.	F	T_e	Gain
1		300 K	10 dB
2	6 dB		30 dB
3	11 dB		30 dB

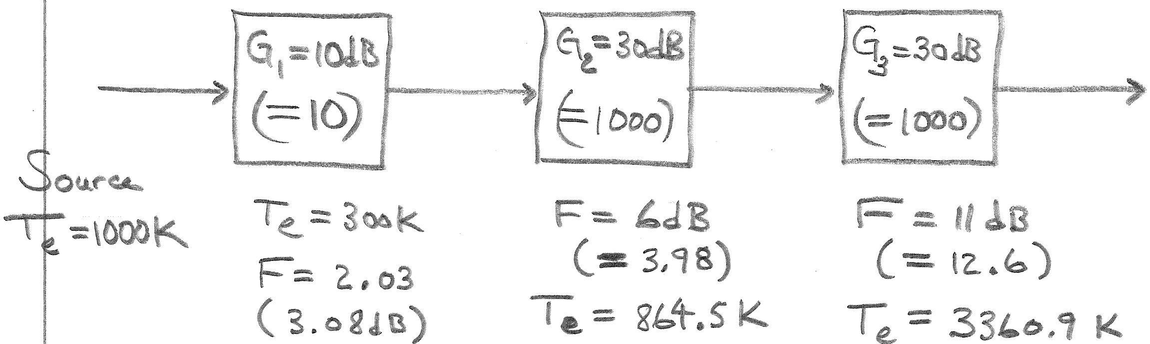
a. Find the noise figure of the cascade.

b. Suppose amplifiers 1 and 2 are interchanged. Find the noise figure of the cascade.

c. Find the noise temperature of the systems of parts (a) and (b).

d. Assuming the configuration of part (a), find the required input signal power to give an output SNR of 40 dB. Perform the same calculation for the system of part (b).

Z+T Problem A.7



Assume that all noise figures are standardized for 290K as discussed in class notes. Then the relationship between noise figure and effective noise temp of amp is

$$T_{amp,e} = (290K)(F_{amp} - 1)$$

$$F_{amp} = \frac{T_{amp,e}}{290K} + 1$$

(a) Noise Figure of the Cascade

$$F_{cascade} = F_1 + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1 G_2}$$

$$= 2.03 + \frac{3.98 - 1}{10} + \frac{12.6 - 1}{10^4}$$

$$= 2.33 \quad (= 3.67\text{dB})$$

(b) Interchange amps 1 and 2 and repeat (a).

$$F_{cascade} = 3.98 + \frac{2.03 - 1}{1000} + \frac{12.6 - 1}{10^4}$$

$$= 3.98 \quad (= 6.00\text{dB})$$

(c) Noise temp of systems of (a), (b).

We already found the equiv. noise temps of the individual elements. To find the system equiv. temps we can use

$$T_{syst,e} = (290K)(F_{system} - 1)$$

$\therefore$

$$T_{syst\_a,e} = (290K)(2.33 - 1)$$
$$= 385.7 \text{ K}$$

$$T_{syst\_b,e} = (290K)(3.98 - 1)$$
$$= 864.2 \text{ K}$$

(d) Assuming the 1000K noise source find the noise power in the output for each of the two systems. Then for output SNR of 40 dB, find the required input signal power.

The one-sided BW is given as 50KHz.

Output noise power $= N_o = k(T_{input} + T_{syst,e}) B G_1 G_2 G_3$

$$N_{o,a} = (1.38\times10^{-23})[1000 + 385.7](50\times10^{3})(10^{7})$$
$$= -50.2 \text{ dBm}$$

$$N_{o,b} = (1.38\times10^{-23})[1000 + 864.2](50\times10^{3})(10^{7})$$
$$= -48.9 \text{ dBm}$$

To have SNR of 40 dB need

$$S_{o,a} = -50.2 + 40 \quad \text{dBm}$$
$$= -10.2\,\text{dBm}$$

Since cascade gain is 70 dB then need

$$S_{in,a} = -10.2\,\text{dBm} - 70\,\text{dB}$$
$$= -80.2\,\text{dBm} \qquad \text{input sig. power}$$

For the system of (b)

$$S_{o,b} = -48.9\,\text{dBm} + 40\,\text{dB}$$
$$= -8.9\,\text{dBm}$$

$$\Rightarrow S_{in,b} = -8.9\,\text{dBm} - 70\,\text{dBm}$$
$$= -78.9\,\text{dBm} \qquad \text{input sig. power}$$

A.12. Given a relay–user link as described in Section A.3 with the following parameters:

Average transmit power of relay satellite: 35 dBW
Transmit frequency: 7.7 GHz
Effective antenna aperture of relay satellite: 1 m^2
Noise temperature of user receiver (including antenna): 1000 K
Antenna gain of user: 6 dB
Total system losses: 5 dB
System bandwidth: 1 MHz
Relay–user separation: 41,000 km

a. Find the received signal power level at the user in dBW.

b. Find the receiver noise level in dBW.

c. Compute the SNR at the receiver in decibels.

d. Find the average probability of error for the following digital signaling methods: (1) BPSK, (2) binary DPSK, (3) binary noncoherent FSK, (4) QPSK.[4]

Z+T Problem A.12

The solution to this problem very closely mirrors the examples given in the Z+T appendix. Also, we skip part (d) since have not yet covered dig. modulations.

The problem only concerns the downlink from the relay satellite to the LEO/aircraft user. The Friis equation modified to include other system losses is

$$P_R = \left(\frac{\lambda}{4\pi d}\right)^2 \frac{P_T G_T G_R}{L_o}$$

where

$$P_T = 35 \text{ dBW}$$

$$\lambda = (3\times10^8 \text{ m/s})/(7.7\times10^9 \text{ s}^{-1}) = 0.039 \text{ m}$$

$$d = 41\times10^3\times10^3 = 4.1\times10^7 \text{ m}$$

$$G_T = 4\pi A_T/\lambda^2$$

$$= 4\pi\left(1/0.039\right)^2 = 8261.9 \quad (= 39.2 \text{ dB})$$

$$G_R = 6 \text{ dB}$$

$$L_o = 5 \text{ dB}$$

$$\Rightarrow \left(\frac{\lambda}{4\pi d}\right)^2 = 5.73\times10^{-21} \quad (= -202.4 \text{ dB})$$

(a) $P_{R,\text{dBW}} = 35 \text{ dBW} + 39.2 \text{ dB} + 6 \text{ dB} - 5 \text{ dB} - 202.4 \text{ dB}$

$$= -127.2 \text{ dBW}$$

(b) Find Receiver Noise Level in dBW

We are given that the equiv. noise temp of the receiver including antenna is 1000K and the system BW is 1MHz.

Then the effective noise power at receiver input is

$$kT_eB = (1.38 \times 10^{-23})(1000)(1 \times 10^{6})$$

$$= 1.38 \times 10^{-14}\ \text{W} \quad (= -138.6\ \text{dBW}).$$

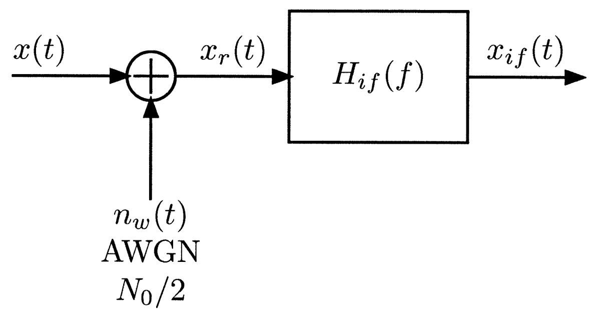
(c) So SNR at receiver

$$= -127.2 - (-138.6)$$

$$= 11.4\ \text{dB}.$$

Correction

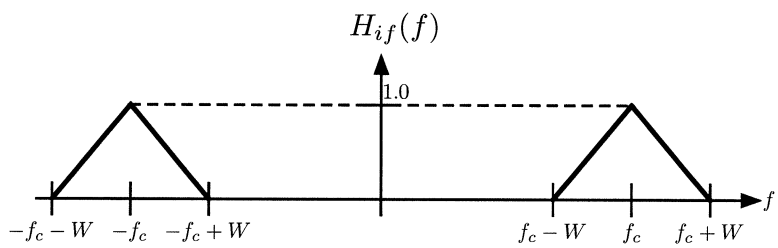
Problem 2. [50 pts. total] The block diagram below shows the front end of an AM DSB communication system.



The input signal is an AM DSB waveform where the message is sinusoidal, i.e.,

$$x(t) = A_c A_m \cos(2\pi f_m t)\cos(2\pi f_c t)$$

($f_m \ll f_c$). We model $x(t)$ as a deterministic power signal for simplicity. The triangularly shaped IF filter $H_{if}(f)$ has bandwidth $2W$ where $0 < W < f_c$. The goal of the problem is to choose W to maximize SNR_T, the ratio of the power of the signal part in $x_{if}(t)$ to the power of the noise part in $x_{if}(t)$.



(a) [10 pts.] Signal:

- Find the signal component in the output of the IF filter, i.e., $x * h_{if}(t)$, as a function of W, $0 < W < f_c$. It is helpful to consider two cases: 1) when $0 < W < f_m$ and 2) when $f_m \le W < f_c$.
- Then find the power of the above signal component in terms of the fixed parameters A_c, A_m, $\Delta \stackrel{\text{def}}{=} f_c - f_m$ and the variable W.
- Make a rough sketch of this power as a function of W, $0 < W < f_c$, and comment.

just use f_m

Problem 2. (cont'd.)

(a) $x(t) = \frac{A_c A_m}{2} \cos 2\pi (f_c - f_m) t + \frac{A_c A_m}{2} \cos 2\pi (f_c + f_m) t$

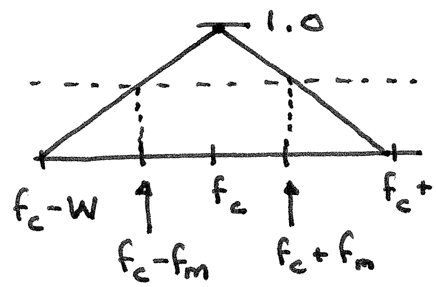
These two tones on either side of f_c at an equal distance f_m away from f_c. Since H_{if} is symmetric about $f = f_c$ the tones undergo identical attenuation in passing through H_{if}.

Case: $0 < W < f_m$

Tones outside of passband $\Longrightarrow$ $x * h_{if}(t) = 0$

Case: $f_m \leq W < f_c$

Tones now inside passband but will be attenuated. The situation is



amplitude level $= 1 - \frac{f_m}{W}$

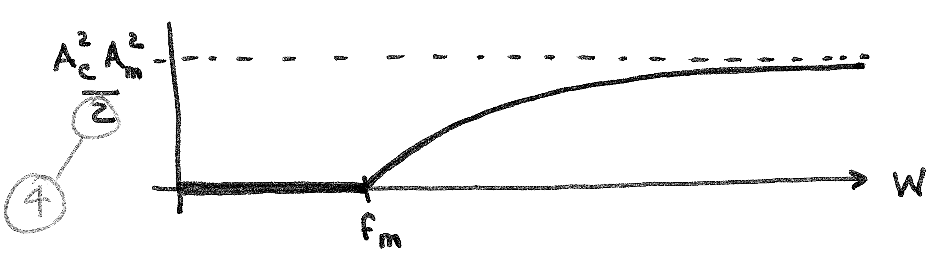
$$\Longrightarrow x * h_{if}(t) = \left(1 - \frac{f_m}{W}\right) x(t)$$

In case $0 < W < f_m$ the power in $x * h_{if} = 0$. In case $f_m \leq W < f_c$ the power is

$$\text{power}[x * h_{if}] = \left(1 - \frac{f_m}{W}\right)^2 \text{power}[x] = \left(1 - \frac{f_m}{W}\right)^2 \frac{A_c^2 A_m^2}{2} \quad 4$$

$$= \left(\frac{W - f_m}{W}\right)^2 \frac{A_c^2 A_m^2}{2} \quad 4$$

Note mistake. Fixes shown in following.



clearly for large W there is attenuation decreasing toward 0.

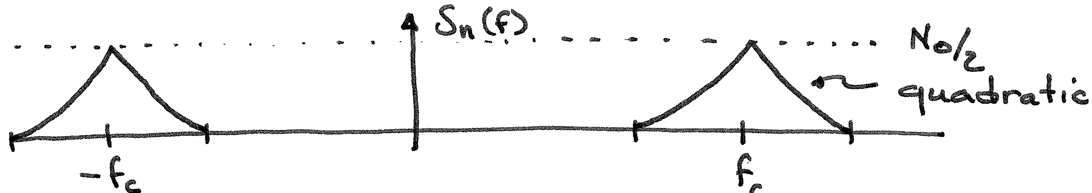
Problem 2. (cont'd.)

(b) [20 pts.] Find the power of the noise component in the output of the IF filter, i.e., $n_w * h_{if}(t)$, as a function of W, $0 < W < f_c$, roughly plot it, and comment.

If $n(t) = n_w * h_{if}(t)$ then $S_n(f) = \frac{N_0}{2} |H_{if}(f)|^2 = \frac{N_0}{2} H_{if}^2(f)$

$\Longrightarrow$ power $[n(t)] = \int_{-\infty}^{\infty} S_n(f)\, df.$

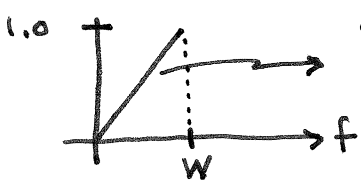
This psd will look roughly like



Everything is symmetric. Hence

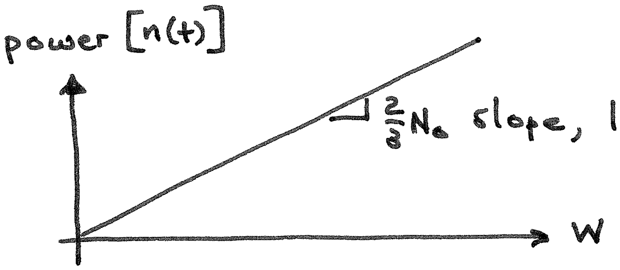
$$\text{power}[n(t)] = 4 \int_{f_c - W}^{f_c} S_n(f)\, df$$

so only need compute area under one of the curves. Also Could clearly change variables and shift:



eqN is f/W $\Longrightarrow$ power $[n(t)] = 4 \int_0^W \frac{N_0}{2} \left(\frac{f}{W}\right)^2 df$

$$= \frac{2N_0}{W^2} \frac{1}{3} f^3 \Big|_0^W = \frac{2}{3} N_0 W$$



$\frac{2}{3} N_0$ slope, linear in W.

Problem 2. (cont'd.)

(c) [5 pts.] Now find SNR_T as defined above and comment on its general behavior as a function of W, $0 < W < f_c$.

$$\mathrm{SNR}_T = \begin{cases} \dfrac{A_c^2 A_m^2}{2 \; 4} \left(\dfrac{W - f_m}{W}\right)^2 \dfrac{3}{2N_o W} & \text{for } 0 < f_m < W < f_c \\ 0 & \text{for } 0 < W < f_m \end{cases}$$

Since noise power grows without bound as $W \to \infty$ while signal power is bounded we know that SNR_T must eventually decrease for large W. Also SNR_T must be zeo @ $W = f_m$.

Hence expect there to be a maximum.

(d) [10 pts.] Find the value W_* maximizing SNR_T as a function of W, $0 < W < f_c$.

For maximization it is enough to examine the factor

$$g(W) \triangleq \frac{(W - f_m)^2}{W^3} \quad \text{for} \quad f_m < W < f_c$$

$$g'(W) = \frac{2(W-f_m)W^3 - 3W^2(W-f_m)^2}{W^6} = \frac{(W-f_m)W^2\left[2W - 3(W-f_m)\right]}{W^6}$$

$$= \frac{(W-f_m)\left[-W + 3f_m\right]}{W^4} \longrightarrow$$ Finite zeros for $W = f_m$ and $W = 3f_m$

Only $W_* = 3f_m$ gives max.

Problem 2. (cont'd.)

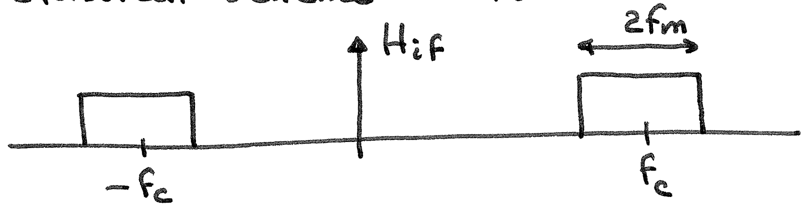
(e) [5 pts.] Let SNR_{T*} be the maximum SNR obtained in (d). What is the degradation in dB of this triangular IF filtering scheme compared to the best one could do with the "classical" rectangular IF filters we examined in class?

$$\text{SNR}_{T*} = \text{SNR}_T(W_* = 3f_m)$$

$$= \frac{A_c^2 A_m^2}{2} \cdot \frac{4}{9} \cdot \frac{3}{2N_0 \cdot 3f_m} = \frac{1}{9} \frac{A_c^2 A_m^2}{N_0 f_m}$$

4 18

The best classical scheme would use



The signal would come through unattenuated so its output power would be

$$\frac{A_c^2 A_m^2}{2}$$

4

and the noise power at the output would be

$$2 \cdot \frac{N_0}{2} \cdot 2f_m = 2N_0 f_m$$

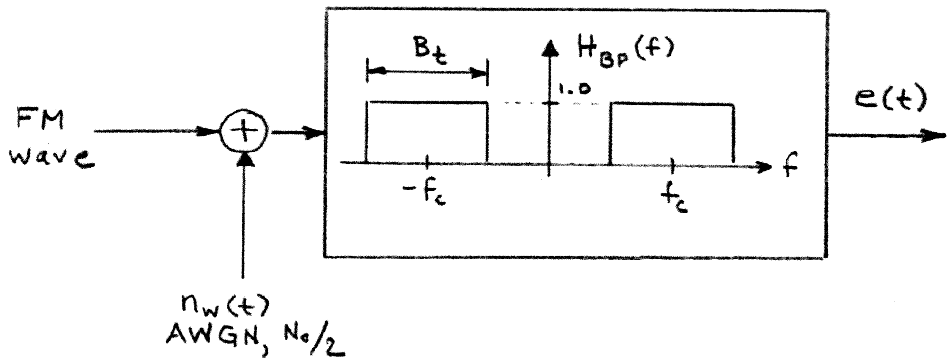
$$\text{SNR}_{T,\text{classical}} = \frac{A_c^2 A_m^2}{4N_0 f_m}$$

8

$$\Rightarrow \frac{\text{SNR}_{T*}}{\text{SNR}_{T,\text{class}}} = \frac{4}{9} \longrightarrow -3.52\,\text{dB}$$

$\frac{8}{18} = \frac{4}{9}$ so ratio does not change.

[Fall 2007 Final Exam] The model for analysis of FM discriminator detection in AWGN assumes a receiver front end consisting of a bandpass filter coming before the discriminator. This front end is shown in the figure below. The BW of the bandpass filter is B_t and is set equal to a value needed to pass the FM wave with minimal distortion.



(a) The signal $e(t)$ at the output of the bandpass filter is the sum of the FM wave of power $P_T = A_c^2$ and the filtered noise $n(t) = [h_{BP} * n_w](t)$. Sketch the power spectral density $S_{n,n}(f)$ of the filtered noise. Compute the power in the noise $n(t)$ and write down the SNR at the output of the bandpass filter.

(b) Let the in-phase/quadrature expansion of $n(t)$ be

$$n(t) = \sqrt{2}n_I(t)\cos(2\pi f_c t) - \sqrt{2}n_Q(t)\sin(2\pi f_c t)$$

where $n_L(t) = n_I(t) + jn_Q(t)$ is the complex envelope of $n(t)$. Find and sketch the power spectral densities of $n_L(t)$, $n_I(t)$, and $n_Q(t)$. Compute the power in each of these random processes. For the scenario given here, what is the cross-correlation function between the in-phase and quadrature components of the noise $n_I(t)$ and $n_Q(t)$?

(c) In the class notes we derived the following expression for the SNR at the output of a discriminator demodulator

$$\text{SNR}_D = K\left(\frac{B_t}{W}\right)^2\left(\frac{P_T}{N_0 W}\right).$$

In the above equation W is the baseband bandwidth of the message used to modulate the FM wave and B_t/W is the bandwidth expansion factor.

In commercial FM broadcast the maximum frequency deviation of an FM wave is limited to 75 kHz by FCC regulation. The message bandwidth is $W = 15$ kHz. Using Carson's rule find the transmission bandwidth B_t. For these parameters find the factor by which demodulated SNR is improved relative to baseband transmission (express both as a ratio and in dB). You may assume $K = 1$ for simplicity.

(d) As mentioned in class the expression for SNR_D would seem to indicate that the improvement over baseband transmission can be made arbitrarily large by increasing the transmission bandwidth. However, the large SNR assumption used to derive the expression will eventually break down. The class analysis was based on the assumption that $A_c \gg |n_L(t)|$. Recall that

$$R = |n_L(t)| = \sqrt{n_I^2(t) + n_Q^2(t)}$$

is a Rayleigh random variable with pdf

$$f_R(r) = \frac{r}{\sigma^2}e^{-r^2/2\sigma^2} \quad \text{for } r \geq 0$$

where $\sigma^2 = \text{Power}\{n_I(t)\}$. Suppose that we define the high SNR region to be the case where

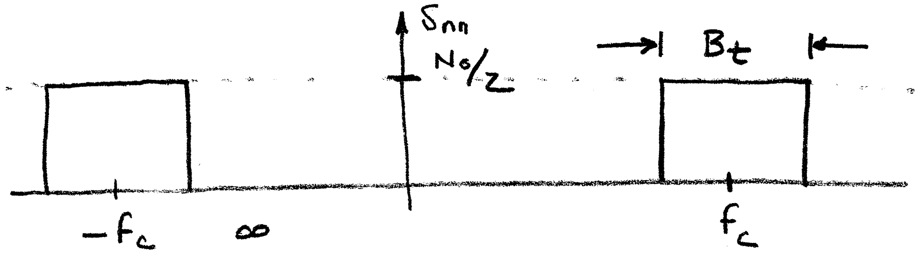
$$\Pr\{|n_L(t)| \leq 0.1A_c\} \geq 0.9.$$

Use this definition of high SNR to find a lower bound on $P_T/(N_0W)$ above which the class expression for SNR_D holds. The lower bound will be of the form

$$\text{some simple function of } \left(\frac{B_t}{W}\right) \leq \frac{P_T}{N_0W}.$$

(e) Evaluate the lower bound for the broadcast FM parameters of Part (c).

(a) $n(t) = h_{BP} * n_W(t) \iff S_{nn}(f) = |H_{BP}(f)|^2 N_0/2$



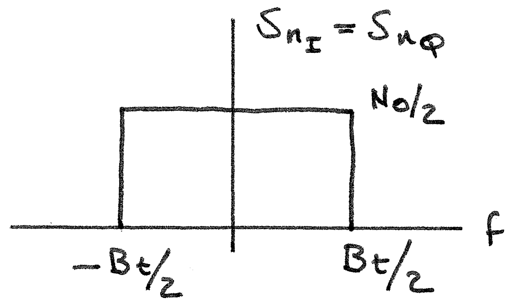
$$\text{Power}\{n(t)\} = \int_{-\infty}^{\infty} S_{nn}(f)\, df = 2 \cdot B_t \cdot \frac{N_0}{2} = B_t N_0$$

$$\text{SNR @ BP output} = \frac{\text{power}\{\text{FM wave}\}}{\text{power}\{n(t)\}} = \frac{A_c^2}{B_t N_0}$$

(b) From class

$$S_{n_I}(f) = S_{n_Q}(f) = \frac{1}{2} \text{LPF}\left\{ S_n(f-f_c) + S_n(f+f_c) \right\}$$

where LPF has a bandwidth to pass baseband components and reject carrier components. Using the assumed symmetry of S_n about f_c in the positive frequencies



and $S_{n_I n_Q}(f) \equiv 0$ for all f. For $n_L(t) \triangleq n_I(t) + j n_Q(t)$ we calculate the autocorrelation

$$R_{n_L}(\tau) = E\left\{ n_L(t+\tau)\, n_L^*(t) \right\}$$

$$= E\left\{ \left[n_I(t+\tau) + j n_Q(t+\tau) \right] \left[n_I(t) - j n_Q(t) \right] \right\}$$

$$= R_{n_I}(\tau) - j R_{n_I n_Q}(\tau) + j R_{n_I n_Q}(-\tau) + R_{n_Q}(\tau)$$

Since $S_{n_I n_Q}(f) \equiv 0 \ \forall f$ have $R_{n_I n_Q}(\tau) \equiv 0 \ \forall \tau$.

$$\Rightarrow R_{n_L}(\tau) = R_{n_I}(\tau) + R_{n_Q}(\tau)$$
$$= 2R_{n_I}(\tau)$$
$$\updownarrow$$
$$S_{n_L}(f) = 2S_{n_I}(f).$$

$$\therefore \text{power}\{n_I\} = \text{power}\{n_Q\} = \frac{N_0 B_t}{2}$$
$$\text{power}\{n_L\} = N_0 B_t$$

(c) Given $SNR_D = \underbrace{K\left(\frac{B_t}{W}\right)^2}_{\text{improvement factor}} \frac{P_T}{N_0 W}$

$\Delta f_{max} = 75\,KHz$, $W = 15\,KHz$. The Carson rule bandwidth formula would give

$$B_t = 2(75+15) = 180\ KHz$$

with $K=1$

$$\text{Imp. Factor} = \left(\frac{B_t}{W}\right)^2 = \left(\frac{180}{15}\right)^2 = 12^2 = 144$$

$$\left(\text{in dB} = 10\log_{10} 144 \approx 21.6\ dB\right)$$

(d) $R = |n_L(t)|$ $\quad f_R(r) = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2}$ $\quad r \geq 0$ where

$$\sigma^2 = \text{Power}\{n_I(t)\} = N_0 B_t/2$$

$$\Pr\{R \leq 0.1 A_c\} = \int_0^{0.1 A_c} \frac{r}{\sigma^2} e^{-r^2/2\sigma^2}\, dr \geqslant 0.9$$

Evaluating the integral let $u = r^2/2\sigma^2 \Rightarrow du = \dfrac{2r\,dr}{2\sigma^2}$

$$= \frac{r}{\sigma^2}\,dr$$

$$\therefore \int_0^{0.1A_c} \frac{r}{\sigma^2} e^{-r^2/2\sigma^2}\,dr = \int_0^{(0.1A_c)^2/2\sigma^2} e^{-u}\,du = -e^{-u}\Big|_{u=0}^{0.01A_c^2/2\sigma^2}$$

$$= 1 - e^{-0.01A_c^2/2\sigma^2} \geq 0.9$$

Now $\dfrac{0.01\,A_c^2}{2\sigma^2} = \dfrac{0.01\,P_T}{N_0 B_t}$

$$1 - 0.9 = 0.1 \geq e^{-0.01 P_T/(N_0 B_t)}$$

$$\ln(0.1) \geq -0.01\,\frac{P_T}{N_0 B_t}$$

$$-100 \ln(0.1) \leq \frac{P_T}{N_0 B_t}\,\frac{W}{W} = \left(\frac{W}{B_t}\right)\frac{P_T}{N_0 W}$$

$$\frac{P_T}{N_0 W} \geq -\left(\frac{B_t}{W}\right) 100 \ln(0.1) = \underbrace{\left(\frac{B_t}{W}\right) 100 \ln(10)}_{\text{lower bound.}}$$

(e) $B_t/W = 12$

$$\frac{P_T}{N_0 W} \geq 12 \cdot 100 \ln(10) \approx 2763 \quad (\text{i.e. } 34.4\text{ dB}).$$

Note: In FM the threshold for high SNR is usually given in terms of the IF SNR (sometimes called carrier-to-noise ratio). From (a)

$$\frac{A_c^2}{B_t N_o} = \frac{P_T}{B_t N_o} = \frac{P_T}{N_o W} \cdot \frac{W}{B_t}$$

$$\therefore \quad \frac{P_T}{N_o B_t} \geq 100 \ln 10 = 230.26 \quad \left(\text{ie } 23.6 \text{ dB}\right)$$

And note that each doubling of the transmission BW B_t decreases the IF SNR by 3 dB.