Name: _____

EE-602
Exam II
March 1, 2006

140 Point Exam

INSTRUCTIONS

This is a closed book, closed notes exam. You are permitted only a calculator. Work patiently, efficiently, and in an organized manner clearly identifying the steps you have taken to solve each problem. Your grade for each problem depends on the completeness, organization and clarity of your work as well as the resulting answer. All students are expected to abide by the usual ethical standards of the university, i.e., your answers must reflect only your own knowledge and reasoning ability. On and off campus students are not communicate with each other about the exam until after it is returned or pay the penalty of an F in the course. Any communication about this exam with off campus students will result in an F.

There are a total of **16 pages** and **7 problems**.

Good luck.

1. **(23 points)** **(a)** **(10 pts)** Derive the controllable canonical state model (matrix) form of the differential equation

$$\dddot{y}(t) + a_1 \ddot{y}(t) + a_2 y(t) = b_2 u(t) + b_1 \dot{u}(t) + b_0 \ddot{u}(t)$$

(b) **(8 pts)** Derive the observable canonical state model (matrix) form of the above differential equation.

(c) **(5 pts)** Determine the state transformation matrix T between the two state models.

2. **(15 points)** Consider the state dynamics

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} u$$

Find state feedback $u \leftarrow Fx + u$ so that the eigenvalues $\{-1, -2, 1, 2\}$ are assigned to the system in such a way as to eliminate all interaction between the 2x2 diagonal blocks of A+BF. Be sure to show the form of A+BF. Credit depends strongly upon your approach. ☺

3. **(25 pts)** Consider the discrete time state dynamics

$$x(k+1) = A(k)x(k) + B(k)u(k)$$

(a) (13 pts) Derive a set of equations for driving x(0) to x(4). Put in matrix form.

(b) (8 pts) Suppose

$$B(k) = \begin{bmatrix} k-1 \\ 1-k \end{bmatrix} \qquad \text{and} \qquad A(k) = \begin{bmatrix} 1 & 1 \\ 1-k & k-1 \end{bmatrix}$$

Compute the minimum energy solution to the state control problem when $x(4) = \begin{bmatrix} 2 & -2 \end{bmatrix}^T$ and $x(0) = 0$.

(c) (4 pts) Characterize the set of all possible solutions.

4. (22 pts) A discrete time system is given by

$$x(k+1) = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$$

(a) (8 pts) Find an expression for $A^k = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}^k$.

(b) (14 pts) If $u(k) = (0.5\lambda)^{-k}$ find the zero-state response $x(n)$ and evaluate at $n = 4$.

5. (26 pts) Consider the linear time varying state model

$$\dot{x}(t) = Ax(t)$$
$$y(t) = C(t)x(t)$$

where A is 3×3 and $C(t)$ is 2×3. Your mission is to compute $x(t)$ from output measurements, which due to budget constraints imposed by the dean and implemented by the school head, have caused a reduction of sensor information. Your only measurement data is $\dot{y}(t)$ and $\ddot{y}(t)$, the change and rate of change caused by the budget reduction.

(a) (12 pts) Develop a set of equations using ONLY $\dot{y}(t)$ and $\ddot{y}(t)$ for computing $x(t)$ for the above equations, and only the above equations. Put in matrix form. (Note, you must determine the proper measurements and numbers of equations. Specify the dimensions of each matrix in your matrix equation.)

(b) (4 pts) Unfortunately, the budget reductions implemented by the school head have lead to the purchase of sensors form the notorious company, Inferior Sensors Ltd, resulting in noisy measurement data and inconsistent equations. Under what conditions and how can one obtain the best least squares solution?

(c) (10 pts) Your inferior sensor measures $\dot{y}(1) = \begin{bmatrix} 2 & 2 \end{bmatrix}^T$ and $\ddot{y}(1) = \begin{bmatrix} -2 & 2 \end{bmatrix}^T$ Fill in the equations

of part (a) for finding and solving for $x(1)$ when $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $C(t) = \begin{bmatrix} 2t-2 & 2t-2t^2 & 0 \\ 0 & t & -t \end{bmatrix}$.

Then find the BEST estimate for $x(1)$.

4. (15 pts) The time-varying state dynamics

$$\dot{x}(t) = Ax(t) + B(t)u(t)$$

has solution

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-q)} B(q)u(q)dq$$

Given that $x(T^-) = 0$, determine what states $x(T^+)$ are reachable using the input $u(t) = \xi_0 \delta(t) + \xi_2 \ddot{\delta}(t)$, for $\xi_i \in R^m$ and A is $n \times n$. State the dimensions of all matrices involved in your answer? Develop equations which show what subspace and its dimension in which $x(T^+)$ lives? Recall the sifting property:

$$\int_{T^-}^{T^+} e^{A(T-q)} B(q) \delta^{(i)}(q-T) dq = (-1)^i e^{AT} \left[\frac{d^i}{dq^i} e^{-Aq} B(q) \right]_{q=T}$$

7. (14 pts) A state model in polytopic form is given by the equations

$$\dot{x}(t) = A(x,t)x(t) + B(x,t)u(t)$$

Develop the best set of conditions on $A(x,t)$, $B(x,t)$, $u(t)$ etc. which will guarantee the existence of a unique solution.

Name: _____

EE-602
Exam II
October 20, 2005

132 Point Exam

INSTRUCTIONS

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There are a total of **16 pages** and **7 problems**.

Good luck.

1. (20 pts) Consider the state dynamics

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(a) (5 pts) Compute e^{At}.

(b) (15 pts) Suppose $x(2) = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$. Find $x(0)$ if $u(t) = 1^+(t)$. Your approach is critical to getting more than half credit.

SOLUTION 1. (a) A is nilpotent of order 2. Therefore $e^{At} = \begin{bmatrix} 1+t & -t \\ t & 1-t \end{bmatrix}$.

(b) $e^{At}B = \begin{bmatrix} 1+2t \\ 2t-1 \end{bmatrix}$. $e^{A(t-q)}B = \begin{bmatrix} 1+2t-2q \\ 2t-2q-1 \end{bmatrix}$. $\int_2^0 e^{-Aq} Bdq = \int_2^0 \begin{bmatrix} 1-2q \\ -1-2q \end{bmatrix} dq = \begin{bmatrix} q-q^2 \\ -q-q^2 \end{bmatrix}_2^0 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$..

$e^{A(0-2)}x(2) = e^{A(-2)}x(2) = \begin{bmatrix} 1+t & -t \\ t & 1-t \end{bmatrix}_{t=-2} x(2) = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Therefore

$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

2. (15 points) Derive the controllable canonical state model (matrix) form of the differential equation

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_3 u + b_2\dot{u} + b_1\ddot{u} + b_0\dddot{u}$$

SOLUTION 2. Define the auxiliary equation as

$$\dddot{\hat{y}} + a_1\ddot{\hat{y}} + a_2\dot{\hat{y}} + a_3\hat{y} = u$$

Define state variables as $x_1 = \hat{y}$, $x_2 = \dot{\hat{y}} = \dot{x}_1$, $x_3 = \ddot{\hat{y}} = \dot{x}_2$, in which case

$\dot{x}_3 = \dddot{\hat{y}} = u - a_1\ddot{\hat{y}} - a_2\dot{\hat{y}} - a_3\hat{y} = u - a_1 x_3 - a_2 x_2 - a_3 x_1$. By linearity

$$y = b_3\hat{y} + b_2\dot{\hat{y}} + b_1\ddot{\hat{y}} + b_0\dddot{\hat{y}}$$
$$= b_3 x_1 + b_2 x_2 + b_1 x_3 + b_0\left(u - a_1 x_3 - a_2 x_2 - a_3 x_1\right)$$
$$= (b_3 - b_0 a_3)x_1 + (b_2 - b_0 a_2)x_2 + (b_1 - b_0 a_1)x_3 + b_0 u$$

Thus

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$

$$y = \begin{bmatrix} (b_3 - b_0 a_3) & (b_2 - b_0 a_2) & (b_1 - b_0 a_1) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [b_0]u$$

3. (15 points) Consider the state dynamics

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \end{bmatrix}x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}u$$

(a) What is the characteristic polynomial of the A-matrix.

(b) Find state feedback $u \leftarrow Fx + u$ so that the eigenvalues $\{-1 \pm j, -1 \pm j\}$ are assigned to the system in such a way as to eliminate interaction between the 2x2 blocks of A.

SOLUTION 3. Observe that if $F = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 \\ f_5 & f_6 & f_7 & f_8 \end{bmatrix}$, then

$$A + BF = \begin{bmatrix} 0 & 1 & 0 & 0 \\ f_1 & f_2 & f_3 & f_4 \\ 0 & 0 & 0 & 1 \\ 1+f_5 & 2+f_6 & f_7 & f_8 \end{bmatrix}$$

In order to eliminate interaction we set $f_3 = 0$, $f_4 = 0$, $f_5 = -1$ and $f_6 = -2$. Thus

$$A + BF = \begin{bmatrix} 0 & 1 & 0 & 0 \\ f_1 & f_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & f_7 & f_8 \end{bmatrix}$$

The characteristic polynomial for each 2x2 block must be $\pi_{2\times2}(\lambda) = \lambda^2 + 2\lambda + 2 = \lambda^2 - f_i\lambda - f_{i+1}$.

Therefore $f_1 = f_2 = f_7 = f_8 = -2$.

4. (30 pts) Consider the linear time varying state model

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

where A is 3×3, B is 3×2, $C(t)$ is 2×3, and $D(t)$ is 2×2.

(a) (10 pts) Develop a set of equations using ONLY $y(t)$ and $\dot{y}(t)$ for computing $x(t)$. Put in matrix form. (Note, you must determine the proper measurements and numbers of equations. Specify the dimensions of each matrix in your equation.)

(b) (4 pts) What specific conditions allow for a unique solution.

(c) (8 pts) It is known that $y(t) = \begin{bmatrix} t^2 - t^3 & t + t^2 \end{bmatrix}^T$. Assuming that $u(t) = 0$, fill in the equations of

part (a) for finding and solving for $x(1)$ when $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $C(t) = \begin{bmatrix} t-1 & 0 & 0 \\ 0 & t & 1 \end{bmatrix}$. Then solve for

$x(1)$ by the easiest way you can see. It is not necessary to compute any inverses at all.

(d) (8 pts) Now find a left inverse of the "R-matrix"? Is it unique? Why or why not?

Solution 4. (a) $y(t) = C(t)x(t) + D(t)u(t)$

$\dot{y}(t) = \dot{C}(t)x(t) + C(t)\dot{x}(t) + D(t)\dot{u}(t) + \dot{D}(t)u(t)$

$\quad = \left(\dot{C}(t) + C(t)A\right)x(t) + C(t)Bu(t) + D(t)\dot{u}(t) + \dot{D}(t)u(t)$
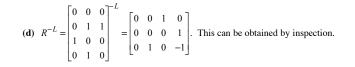
In matrix form, we have

$$\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} C(t) \\ \dot{C}(t) + C(t)A \end{bmatrix} x(t) + \begin{bmatrix} D(t) & 0 \\ C(t)B + \dot{D}(t) & D(t) \end{bmatrix}$$

(b) Consistent equations and full column rank.

(c)

$$\begin{bmatrix} 0 \\ 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(1) \Rightarrow x(1) = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$$

(d) $R^{-L} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{-L} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$. This can be obtained by inspection.

5. (20 pts) The time-varying state dynamics

$$\dot{x}(t) = Ax(t) + B(t)u(t)$$

has solution

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-q)}B(q)u(q)dq$$

Given that $x(T^-) = 0$, determine what states $x(T^+)$ are reachable using the input $u(t) = \xi\ddot{\delta}(t)$, for $\xi \in R^m$. Hint: What are the dimensions of the matrices involved in your answer? What space (and of what dimension) must $x(T^+)$ live in? Recall the sifting property:

$$\int_{T^-}^{T^+} e^{A(T-q)}B(q)\delta^{(i)}(q-T)dq = (-1)^i e^{AT}\left[\frac{d^i}{dq^i}e^{-Aq}B(q)\right]_{q=T}$$

Solution 5.

$$\int_{T^-}^{T^+} e^{A(T-q)}B(q)\ddot{\delta}(q-T)dq = (-1)^2 e^{AT}\left[\frac{d^2}{dq^2}e^{-Aq}B(q)\right]_{q=T}$$

Now

$$\frac{d}{dq}e^{-Aq}B(q) = -Ae^{-Aq}B(q) + e^{-Aq}\dot{B}(q)$$

Thus

$$\left[\frac{d^2}{dq^2}e^{-Aq}B(q)\right]_{q=T} = \left[\frac{d}{dq}\left(-Ae^{-Aq}B(q) + e^{-Aq}\dot{B}(q)\right)\right]_{q=T}$$

$$= A^2 e^{-AT}B(T) - Ae^{-AT}\dot{B}(T) - Ae^{-Aq}\dot{B}(T) + e^{-AT}\ddot{B}(T)$$

Since A and $\exp(At)$ commute,

$$range[x(T^+)] = col-sp\left[(-1)^2 e^{AT}\left[\frac{d^2}{dq^2}e^{-Aq}B(q)\right]_{q=T}\right] = col-sp\left[A^2 B(T) - 2A\dot{B}(T) + \ddot{B}(T)\right]$$

This is at most m-dimensional. So $x(T^+)$ will live in an m-dimensional subspace of R^n.

6. (17 pts) Consider the discrete time state dynamics

$$x(k+1) = A(k)x(k) + B(k)u(k)$$

(a) Derive a set of equations for driving x(1) to x(4). Put in matrix form.
(b) Suppose

$$B(k) = \begin{cases} \begin{bmatrix} 0 \\ 0 \end{bmatrix} & k \text{ even} \\ \begin{bmatrix} 1 \\ 1 \end{bmatrix} & k \text{ odd} \end{cases} \quad\text{and}\quad A(k) = \begin{cases} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} & k \text{ even} \\ \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} & k \text{ odd} \end{cases}$$

Compute the minimum energy solution to the state control problem when x(4) = x(1) = [2 −2]^T.
(c) Characterize the set of all possible solutions.

SOLUTION 6. (a)

$x(2) = A(1)x(1) + B(1)u(1)$
$x(3) = A(2)x(2) + B(2)u(2) = A(2)A(1)x(1) + A(2)B(1)u(1) + B(2)u(2)$
$x(4) = A(3)x(3) + B(3)u(3) = A(3)A(2)A(1)x(1) + A(3)A(2)B(1)u(1) + A(3)B(2)u(2) + B(3)u(3)$

In matrix form

$$x(4) = A(3)A(2)A(1)x(1) + \begin{bmatrix} B(3) & A(3)B(2) & A(3)A(2)B(1) \end{bmatrix}\begin{bmatrix} u(3) \\ u(2) \\ u(1) \end{bmatrix}$$

(b) and (c)

$$\begin{bmatrix} 2 \\ -2 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} u(3) \\ u(2) \\ u(1) \end{bmatrix}$$

Therefore

$$\begin{bmatrix} -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} u(3) \\ u(2) \\ u(1) \end{bmatrix} \Rightarrow \begin{bmatrix} u(3) \\ u(2) \\ u(1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0.5 & -0.5 \end{bmatrix}\begin{bmatrix} -2 \\ -2 \end{bmatrix} + a\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ a \\ 0 \end{bmatrix}$$

7. (15 pts) Consider the time varying single-input state model

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

where

$$\exp\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} t \right) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

Suppose system input satisfies $u(t) = u(kT)1^+(t)$ for $kT \le t < (k+1)T$, $k = 0, 1, 2, \dots$
Construct the matrices of a discrete time state model

$$z(k+1) = \hat{A}\, z(k) + \hat{B}\, \hat{u}(k)$$

so that $z(k) = x(kT)$ for all integers $k \ge 0$, i.e.,

    (i) Construct the $\hat{A}(k)$-matrix.
    (ii) Construct the $\hat{B}(k)$-matrix.

SOLUTION 7. Note first that

(4 pts) $\quad x\big((k+1)T\big) = e^{AT} x(kT) + \left[\int_{kT}^{(k+1)T} e^{A((k+1)T-q)} B \, dq \right] u(kT)$

Hence

(3 pts) $\quad \hat{A} = \exp\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} T \right) = \begin{bmatrix} \cos(T) & \sin(T) \\ -\sin(T) & \cos(T) \end{bmatrix}$

(8 pts)

$$\hat{B} = \int_{kT}^{(k+1)T} e^{A((k+1)T-q)} B \, dq = \int_{kT}^{(k+1)T} \begin{bmatrix} \cos((k+1)T-q) \\ -\sin((k+1)T-q) \end{bmatrix} dq = \begin{bmatrix} -\sin((k+1)T-q) \\ -\cos((k+1)T-q) \end{bmatrix}_{kT}^{(k+1)T} = \begin{bmatrix} \sin(T) \\ -1 + \cos(T) \end{bmatrix}$$

Name: _____

EE-602
Exam II
October 19, 2004

132 Point Exam

INSTRUCTIONS

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There are a total of 16 pages and 7 problems.

Good luck.

1. (12 pts) Consider the state dynamics

$$\dot{x}(t) = A x(t) + B u(t), \quad x(2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where $A = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $e^{At} = I + A(e^t - 1)$. If $u(t) = e^t \, 1^+(t)$ find x(0).

2. (15 pts) Consider the system $\dot{x}(t) = A x(t) + B u(t)$. It is desired to drive $x(0^-) = [0 \quad 0 \quad 1]^T$ to $x(0^+) = [1 \quad 1 \quad 1]^T$ using an input of the form $u(t) = \xi_0 \delta(t) + \xi_1 \dot{\delta}(t)$. It is known that an svd of the matrix $Q = [B \quad AB]$ is given below the problem statement along with the pseudo inverse of Q.

(a) (2 pts) Define the Moore-Penrose pseudo right inverse of Q.
(b) (6 pts) Find the coefficients ξ_0 and ξ_1.
(c) (4 pts) Find a general solution for the vector $[\xi_0^T \quad \xi_1^T]^T$.
(d) (3 pts) Find an orthogonal basis for $Im[Q] = col - sp[Q]$.

```
»[U,S,V] = svd(Q)
U =
   5.2573e-01        0  -8.5065e-01
        0   1.0000e+00        0
   8.5065e-01        0   5.2573e-01
S =
   1.6180e+00        0        0        0
        0   1.0000e+00        0        0
        0        0   6.1803e-01        0
V =
   8.5065e-01        0  -5.2573e-01        0
        0   1.0000e+00        0        0
   5.2573e-01        0   8.5065e-01        0
        0        0        0   1.0000e+00
```

Further, it is known that

```
Qpseudo-inv =
   1.0000e+00        0   3.1565e-17
        0   1.0000e+00        0
  -1.0000e+00        0   1.0000e+00
        0        0        0
```

3. (15 pts) Consider the linear time varying state model

$$\dot{x}(t) = A x(t) + B u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

where A is 2×2, B is 2×2, $C(t)$ is 2×2, and $D(t)$ is 2×2.

(a) (10 pts) Develop a set of equations for computing $x(t)$ from input-output measurements. Put in matrix form. (Note, you must determine the proper measurements and numbers of equations.)

(b) (5 pts) Under what conditions are the resulting equations solvable for a unique solution. How would one solve the equations, at least theoretically.

4. (20 pts) Consider the discrete time state model

$$x(k+1) = A x(k) + B u(k)$$
$$y(k) = C x(k) + D u(k)$$

From MATLAB you have the data set forth below.

(a) (6 pts) Besides the left inverse given in the MATLAB data, find another left inverse of R (see R below).

(b) (7 pts) If $y(0) = [1 \quad 3]^T$ and $y(1) = [1 \quad 1]^T$ find $x(0)$ assuming the input sequence is zero. Which left inverse gives the unique solution?

(c) (7 pts) If your data is corrupted by a prof-speak obscurity filter to $y(0) = [0 \quad 2]^T$ and $y(1) = [-1 \quad 0]^T$ find a least squares solution for $x(0)$ again assuming the input sequence is zero.

```
»R = [C;C*A]
R =
   1    1
   1   -1
   1    1
   1    1
»[U,S,V]=svd(R)
U =
   5.7735e-01          0  -5.7735e-01  -5.7735e-01
          0  -1.0000e+00           0           0
   5.7735e-01          0   7.8868e-01  -2.1133e-01
   5.7735e-01          0  -2.1133e-01   7.8868e-01
S =
   2.4495e+00          0
          0   1.4142e+00
          0          0
          0          0
V =
   7.0711e-01  -7.0711e-01
   7.0711e-01   7.0711e-01
»Sinv = pinv(S)
Sinv =
   4.0825e-01          0          0          0
          0   7.0711e-01          0          0
»RL1 = V*Sinv*U'
RL1 =
   1.6667e-01   5.0000e-01   1.6667e-01   1.6667e-01
   1.6667e-01  -5.0000e-01   1.6667e-01   1.6667e-01
```

5. (25 pts) Consider a discrete time system $x(k+1) = Ax(k) + Bu(k)$ where $u(k) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} a^{-k} 1^+(k)$,

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

(a) (10 pts) Let $A = TJT^{-1}$. Derive a closed form expression for J^k. Using induction, verify your expression. What then is A^k?

(b) (15 pts) Find the zero-state state-response, x(n) assuming a ≠ 0 and a ≠ 1.

Hint: Recall that

$$\sum_{j=0}^{k-1} \lambda^j = \frac{1 - \lambda^k}{1 - \lambda} \text{ and } \sum_{j=0}^{k-1} j\lambda^{j-1} = \frac{d}{d\lambda}\left(\frac{1 - \lambda^k}{1 - \lambda}\right)$$

6. (22 pts) Consider the state dynamics

$$\dot{x}(t) = A(x,t)x(t) + B(x)f[u(t)]$$

(a) (5 pts) State the Lipschitz condition, precisely.

(b) (12 pts) Develop conditions on $A(x,t)$, $B(x)$, $f(\cdot)$, and $B(x)f[u(t)]$ which are sufficient for the existence of a unique solution. Explain.

7. (23 pts) Let $v_1, \ldots, v_n$ be a basis for R^n. Suppose that n solutions to the differential equation

$$\dot{x}(t) = A(t)x(t), \quad x(t_0) = v_i$$

are given by $\phi_i(t, t_0; v_i)$.

(a) (8 pts) Show that these solutions are independent.

(b) (15 pts) Suppose $\psi(t, t_0, v)$ is a solution to

$$\dot{x}(t) = A(t)x(t), \quad x(t_0) = x_0$$

for some v in R^n. Show that $\psi(t, t_0, v)$ can be expressed as a linear combination of the $\phi_i(t, t_0, v_i)$. Specify the linear combination.

Name: <u>SOLUTIONS</u>

EE-602
EXAM II
OCTOBER 10, 2003

125 Point Exam

INSTRUCTIONS

This is a closed book, closed notes exam. Work patiently, efficiently, and in an organized manner clearly identifying the steps you have taken to solve each problem. Your grade for each problem depends on the completeness, organization and clarity of your work as well as the resulting answer. All students are expected to abide by the usual ethical standards of the university, i.e., your answers must reflect only your own knowledge and reasoning ability.

There are a total of 15 pages and 7 problems.
NO CALCULATORS, NO SCRAP PAPER

Good luck.

Trivia for your parents' parties: (1) Fermat numbers have the form $F_n = 2^{2^n} + 1$. $F_0 = 3$, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$, and $F_4 = 65537$ is the largest known prime Fermat number.
(2) You can't put an OLD head on YOUNG shoulders.

1. (18 pts) (a) (8 pts) Given a nonsingular state transformation Tz = x and the state dynamics

$$\dot{x} = Ax + Bu$$

find $\hat{A}$ and $\hat{B}$ for

$$\dot{z} = \hat{A}z + \hat{B}u$$

in terms of A, B, and T.

(b) (10 pts) If

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

find T so that

$$\hat{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

where λ_1 and λ_2 are the eigenvalues of A, i.e., the roots of the characteristic polynomial of A. Hint: answer is not unique.

Solution. (a) After suitable algebra, $\dot{z} = T^{-1}ATz + T^{-1}Bu = \hat{A}z + \hat{B}u$ and $\lambda_1 = 1$ and $\lambda_2 = 2$.

(b) $T\hat{A} = AT$ Hence $\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$. It follows that

$$\begin{bmatrix} t_{11}\lambda_1 & t_{12}\lambda_2 \\ t_{21}\lambda_1 & t_{22}\lambda_2 \end{bmatrix} = \begin{bmatrix} t_{11} & 2t_{12} \\ t_{21} & 2t_{22} \end{bmatrix} = \begin{bmatrix} t_{21} & t_{22} \\ 3t_{21} - 2t_{11} & 3t_{22} - 2t_{12} \end{bmatrix}$$

Hence, $t_{11} = t_{21}$ and $2t_{12} = t_{22}$. It follows that $T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{11} & 2t_{12} \end{bmatrix} \Rightarrow T = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ will work.

One could also obtain an equivalent solution with $\lambda_1 = 2$ and $\lambda_2 = 1$ or one could solve this in general with λ_1 and λ_2 as variables.

2. (15 points) Derive, along the lines describe in class, the observable canonical form for the following discrete time difference equation:

$$y(k+2) + a_1 y(k+1) + a_0 y(k) = b_0 u(k) + b_1 u(k+1) + b_2 u(k+2)$$

(a) Determine the proper number of state variables.
(b) Define the output equation of the state model.
(c) Complete the derivation explaining what you did.

3. (20 pts) Suppose you want the characteristic polynomial of the feedback system

$$\dot{x} = (A + BF)x$$

to be $\pi_{A+BF}(\lambda) = (\lambda^2 + 2\lambda + 10)(\lambda + 1)$ where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ a & b & c \\ c & b & a \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(a) (5 pts) Write down the structure of F.
(b) (15 pts) Calculate the entries of F.

Solution: (a) $F = \begin{bmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \end{bmatrix}$

(b) $A + BF = \begin{bmatrix} 0 & 1 & 0 \\ a+f_1 & b+f_2 & c+f_3 \\ c+f_4 & b+f_5 & a+f_6 \end{bmatrix}$. There are three simplifying solutions:

(i) make A+BF block diagonal in which case one may choose:

$$F = \begin{bmatrix} -a-10 & -b-2 & -c \\ -c & -b & -a-1 \end{bmatrix}$$

(ii) make A+BF block upper triangular in which case one may choose:

$$F = \begin{bmatrix} -a-10 & -b-2 & f_3 \\ -c & -b & -a-1 \end{bmatrix}$$

for arbitrary f_3.

(iii) ake A+BF block upper triangular in which case one may choose:

$$F = \begin{bmatrix} -a-10 & -b-2 & -c \\ f_4 & f_5 & -a-1 \end{bmatrix}$$

for arbitrary f_4 and f_5.

4. (26 pts) A particular physical process has the linear time varying state model

$$\dot{x}(t) = Ax(t) + B(t)u(t)$$

where $x(t) \in R^2$ and $u(t) \in R$ for each t.

(a) (13 pts) Derive using the sifting property below a set of equations for driving a given $x(T^-)$ to a desired $x(T^+)$ using an impulsive input of the correct form. Specify the form of the input along with the dimension of the coefficients. Recall the sifting property:

$$\int_{T^-}^{T^+} e^{A(T-q)} B(q) \delta^{(i)}(q-T) dq = (-1)^i e^{AT} \left[\frac{d^i}{dq^i} e^{-Aq} B(q) \right]_{q=T}$$

(b) (7 pts) Find a minimum energy impulsive input to drive $x(1^-) = \begin{bmatrix} -1 & 0 \end{bmatrix}^T$ to $x(1^+) = \begin{bmatrix} 9 & 1 \end{bmatrix}^T$ when

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} 0 & t(1-t) \\ 1-t & 0 \end{bmatrix}$$

(c) (6 pts) Characterize the set of all possible solutions to part (b).

Solution:
(a) Note that

$$\int_{T^-}^{T^+} e^{A(T-q)} B(q) \delta^{(i)}(q-T) dq = (-1)^i e^{AT} \left[\frac{d^i}{dq^i} e^{-Aq} B(q) \right]_{q=T} = \begin{cases} B(T) & i=0 \\ AB(T) - \dot{B}(T) & i=1 \end{cases}$$

Further, $u(t) = \xi_0 \delta(t-T) + \xi_1 \delta'(t-T)$, where the coefficients are 2-vectors. Thus

$$\sum_{i=0}^{1} \int_{T^-}^{T^+} e^{A(T-q)} B(q) \xi_i \delta^{(i)}(q-T) dq = B(T)\xi_0 + \left[AB(T) - \dot{B}(T) \right]\xi_1$$

In matrix form

$$x(T^+) - x(T^-) = \begin{bmatrix} B(T) & AB(T) - \dot{B}(T) \end{bmatrix} \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix}$$

(b) From part (a)

$$x(1^+) - x(1^-) = \begin{bmatrix} 10 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \text{ implies } \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 10 \end{bmatrix}$$

(c) By inspection

$$\begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 10 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

where the columns of the 4x2 matrix span the null space of the Q-matrix.

5. (17 pts) Consider the time invariant state dynamics in which T is a nonsingular matrix:

$$x(k+1) = T\begin{bmatrix} 0 & 0 \\ 0 & 0.25 \end{bmatrix}T^{-1}x(k) + T\begin{bmatrix} 1 \\ 1 \end{bmatrix}u(k)$$

(a) (4 pts) Construct an expression for A^k.

(b) (13 pts) Find the zero-state state-response, i.e., find x(9), given the input $u(k) = (0.5)^k 1^+(k)$.

Solution: (a)

$$A^k = T\begin{bmatrix} 0 & 0 \\ 0 & 0.25 \end{bmatrix}^k T^{-1} = T\begin{bmatrix} 0^k & 0 \\ 0 & 0.25^k \end{bmatrix}T^{-1}$$

(b)

$$x(k) = T\sum_{j=0}^{k-1}\begin{bmatrix} 0^j & 0 \\ 0 & 0.25^j \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}0.5^{k-1-j} = T\sum_{j=0}^{k-1}\begin{bmatrix} 0^j \\ 0.25^j \end{bmatrix}0.5^{k-1-j} = 0.5^{k-1}T\begin{bmatrix} 1 \\ \dfrac{1-0.5^k}{1-0.5} \end{bmatrix}$$

6. (16 pts) A discrete time system has the form

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = C(k)x(k) + Du(k)$$

(a) (7 pts) Derive a minimal set of equations for obtaining x(k) from input-output measurements assuming A is 2×2.

(b) (9 pts) Now suppose

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} k-1 & k-1 \\ k-1 & 1-k \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$$

Find $x(1)$ given that $\{u(1)=1, u(2)=2\}$ and $\left\{ y(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, y(2) = \begin{bmatrix} -2 \\ -2 \end{bmatrix} \right\}$.

Solution:

```
»x1 = [1  -3]';
»x2 = A*x1 + B * 1
x2 =
-3
  -1

»y2 = C2*x2+D*2
y2 =
-2
  -2

»y1 = 1*D

y1 =

  1
  0

»Y = [y1;y2];
»U = [1  2]';
»T = [D, zeros(2,1);C2*B, D]
T =
1   0
  0   0
  1   1
  -1   0
```

```
»v = Y-T*U
v =
0

  0
  -5
  -1

»R = [C1;C2*A]
R =
0   0
  0   0
  -2   1
  2   1

»x1 = R\v
x1 =
1.0000e+00
  -3.0000e+00

»RL=pinv(R)

RL =

        0        0  -2.5000e-01   2.5000e-01
  -1.5819e-17        0   5.0000e-01   5.0000e-01
```

7. (20 points) Consider the state dynmics

$$\dot{x} = A(t)f(x) + B(x,t)u(t)$$

(a) State the Lipschitz condition, precisely.

(b) Develop conditions on A(t), f(x), B(x,t), and u(t) which are sufficient for the existence of a unique solution. Explain your reasoning.