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1. Let $E \subseteq \mathbb{R}$. Prove the following are equivalent
   (a) $\forall A \subseteq \mathbb{R}, |A|_e = |A \cap E|_e + |A \cap E^c|_e$
   (b) $\forall \epsilon > 0, \exists O$, an open set such that $E \subseteq O$, and $|O - E|_e < \epsilon$.
   (c) $\forall \epsilon > 0, \exists F$, a closed set such that $F \subseteq E$, and $|E - F|_e < \epsilon$.

2. Let both f and g be functions of bounded variation on $[a, b]$. Prove that $fg, f + g$ are also of bounded variation on $[a, b]$. Suppose $g > \epsilon > 0$ and show f/g is also of bounded variation.

3. Let $\{r_n\}$ be an enumeration of the rationals. Show that the function
$$f(x) = \sum_{\{n:\ r_n < x\}} \frac{1}{2^n}$$
is strictly increasing and $\mathbb{Q} = \{x : f \text{ is discontinuous at } x\}$.

4. For any sequence $\{E_n\}$ of sets and A also a set, prove
$$A - \limsup E_n = \liminf(A - E_n)$$
$$A - \liminf E_n = \limsup(A - E_n).$$

5. Let χ_n be the indicator function for $E_n, n \in \mathbb{N}$ and $A = \limsup E_n, B = \liminf E_n$. Show that
$$\chi_A = \limsup_{n\to\infty} \chi_n,$$
$$\chi_B = \liminf_{n\to\infty} \chi_n.$$

6. Let a_n, b_n be sequences in $\mathbb{R}$. Compare the two quantities
$$\limsup(a_n b_n),$$
$$(\limsup a_n)(\limsup b_n)$$
and provide necessary proofs or counter examples to support your statements. Next prove $\liminf -a_n = -\limsup a_n$.

7. Let $E \subset [0, 1]$, $f : E \to \mathbb{R}$. Show that f is uniformly continuous on E if and only if there is a sequence of polynomials $\{p_n(x)\}$ such that $p_n \to f$ uniformly on E.

8. For $f : [0, 1] \to \mathbb{R}$, let $V(f; [0, 1])$ be the variation of f over $[0, 1]$. If $f_n \to f$ pointwise on $[0, 1]$, show that $V(f; [0, 1]) \le \liminf V(f_n; [0, 1])$.

9. Given $S \subseteq \mathbb{R}$; for every finite subset $\{S_1, \ldots, S_n\} \subset S$, we have $|S_1 + S_2 + \cdots + S_n| \le 1$. Show that S is countable.

10. Let $\alpha_n \in \mathbb{R}$ with $\sum |\alpha_n| < \infty$. If $\{r_n\}$ is an enumeration of the rationals in $I_0 = [0, 1]$, show that

$$\sum \frac{\alpha_n}{\sqrt{|x - r_n|}}$$

converges absolutely for a.e. $x \in I_0$.