22 MARCH 2012 MA375 Pg 1

Graph Theory, cont.

DEF: PATH

$n \in \mathbb{N}$, G: undirected path.

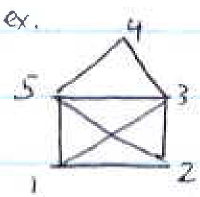
For $u, v \in G$, a path of length n from u to v in G is a sequence of edges $e_1, e_2, \ldots, e_n$ in G for which there exists ~~vertexes~~ a sequence of vertices $x_0, x_1, \ldots, x_n$ in G ($x_0 = u$, $x_n = v$) such that the endpoint of e_i is x_{i-1}, x_i

→ notice that e_i's and x_i's can be repeated.

→ a path is simple if edges, e_i, are not repeated.

→ a path is a circuit if $u = v$.

ex.



1 − 3 − 4 − 5 − 2

is a path of length 4 from 1 to 2.

Q: How many paths from 1 to 2 of length n?

if	$n=1$	1 path	1−2		
	$n=2$	2 path	1−5−2		
	$n=3$	7 path	1−2−1−2	1−2−3−2	1−2−5−2
			1−3−5−2	1−3−1−2	
			1−5−3−2	1−5−1−2	
	$n=4$	?			

Observation: such edge DNE

for $n=2$ 1−1−2 ✗ 1−2−2 ✗ 1−3−2 ✓ 1−4−2 ✗ 1−5−2 ✓

1−i−2 is a path ⟺ 1 & i is adjacent, i + 2 are adjacent

Looking at the adjacent matrix

$$= \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & \phi & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

condition adj 1 and 2 $\iff A_{1i} \neq 0$ and $A_{i2} \neq 0$

then we only see 2 possible choices satisfying such requirement

There are $A_{1i} * A_{i2}$ paths of the form $1-i-2$

There are $\underbrace{\sum_{i=1}^{5} A_{1i} A_{i2}}_{(A^2)_{12}}$ paths of length 2 from 1 to 2

$n = 3$

$1-1-?-2$, $1-\underbrace{2-?-2}$, $1\underbrace{3-?-2}$, $1-4-?-2$, $1-\underbrace{5,-?-2}$

$A_{11} \cdot (A^2)_{12} + A_{12} \cdot (A^2)_{22} + A_{13} (A^2)_{32}$ $A_{14} (A^2)_{42}$ $A_{15} \cdot (A^2)_{52}$

There are $\underbrace{\sum_{i=1}^{5} A_{1i} (A^2)_{i2}}_{(A^3)_{12}}$ paths of length 3 from 1 to 2

Thm 1 Let G be a undirected graph with adjacent matrix A. The number of paths of length n from v_i to v_j is $(A^n)_{ij}$.

Remark last time, for directed graph e.g.

the adjacent matrix is $\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$

Correction: Ignore the ~~negative on~~ -1 ~~to apply~~

$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

Definition

In a directed graph, a directed path from u to v is a path u to v $(x_0 e_1 x_1 e_2 \ldots x_{n-1} e_n x_n)$ such that $e_i : x_{i-1} \to x_i$

Thm 1' G: directed graph w/ adj Mat A. Then the number of directed path of length n is $(A^n)_{ij}$.

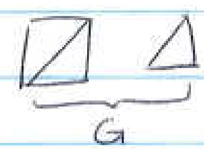
Definition

An undirected graph is <u>connected</u> if there exists a path between every pair of distinct vertices. Otherwise, it is disconnected.

A directed graph is <u>weakly connected</u> if underlying undirected graph is connected.

It is <u>strongly connected</u> if there is a direct path from a to b for any $a, b \in G$ (a, b need not be distinct)

A <u>connected component</u> of a graph G is a connected subgraph of G that is not a proper subgraph of another connected subgraph of G



<u>Question</u> How to test connectivity?

Suppose G has n vertices. v_1, v_2, v_n

If v_i and v_j $(i \neq j)$ can be connected, then they can be connected by a simple path

In particulary, the can be connected by a path of length $\leq n-1$

→ $(A^{\ell})_{ij}$ is the number of paths of length ℓ from v_i to v_j

→ $\sum_{\ell=1}^{n-1} (A^{\ell})_{ij}$ is the number of paths of length $\leq n-1$ from v_i to v_j

Corollary: undirected G is connected if $\left(\sum_{\ell=1}^{n-1} A^{\ell}\right)_{ij} \neq 0$ for all $i \neq j$.

Remark: ① Only have to check $\left(\sum_{\ell=1}^{n-1} A^{\ell}\right)_{1j} \neq 0$ for $j = 2, \dots, n$

① In particular check A, $A+A^2$, $A+A^2+A^3+\dots$

Def: Complementary Graph

G = simple graph

The complementary graph $\overline{G}$ of G is a graph that has same vertices as G

Two vertices are adj in $\overline{G}$ $\Leftrightarrow$ they are not ad in G

e.g.  G compet graph $\overline{G}$