$A \cup B = \{c \in S : c \in A \text{ or } c \in B\}$

$A \cap B = \{c \in S : c \in A \text{ and } c \in B\}$

$\bar{A} = \{a' \in S : a' \notin A\}$

$A - B = \{a \in A : a \notin B\} = A \cap \bar{B}$

<u>Ex</u> $S = \{1, 2, \ldots, 10\}$, $A = \{1, 2, 3, 4, 5\}$

$B = \{b \in S : b \text{ is even}\}$

Find the following: $A \cup B$, $A \cap B$, $\bar{A}$, $\bar{B}$, $A - B$

$B = \{2, 4, 6, 8, 10\}$, $A = \{1, 2, 3, 4, 5\}$

$A \cup B = \{1, 2, 3, 4, 5, 6, 8, 10\}$

$A \cap B = \{2, 4\}$

$\bar{A} = \{6, 7, 8, 9, 10\}$

$\bar{B} = \{1, 3, 5, 7, 9\}$

$A - B = A \cap \bar{B} = \{1, 3, 5\}$

<u>Properties of Set Operations</u>

Let A, B, C be sets in a space S

1) $A \cup B = A$ if $B \subset A$

2) $A \cap B = A$ if $A \subset B$

3) $A \cap S = A$, $A \cup S = S$

4) $A \cap \emptyset = \emptyset$, $A \cup \emptyset = A$

Commutativity: $A \cup B = B \cup A$
$A \cap B = B \cap A$

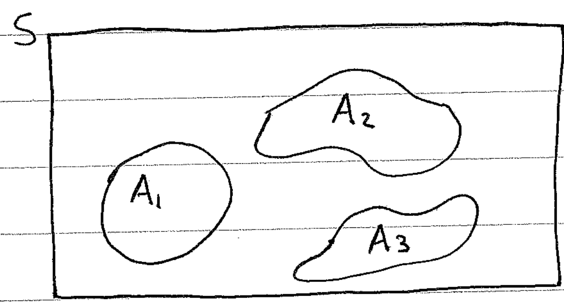
Associativity : $A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C$
$A \cap (B \cap C) = (A \cap B) \cap C = A \cap B \cap C$

Distributivity: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

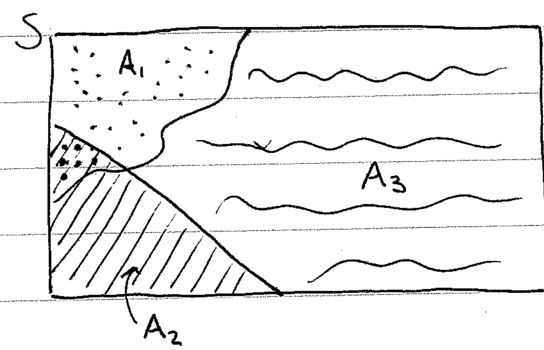
De Morgan's Law : $\overline{A \cup B} = \bar{A} \cap \bar{B}$
$\overline{A \cap B} = \bar{A} \cup \bar{B}$

Def: We say sets $A_1, \dots, A_n$ are **mutually exclusive** (disjoint) if $A_i \cap A_j = \emptyset$ for $i \neq j$

Def: We say sets $A_1, \dots, A_n$ are **collectively exhaustive** if $A_1 \cup \dots \cup A_n = S$



A_1, A_2, A_3 mutually exclusive



A_1, A_2, A_3 collectively exhaustive

Ex Let $S = \{1,2,3,4,5,6\}$

$A = \{1,2,3\}$, $B = \{3,4\}$, $C = \{5,6\}$

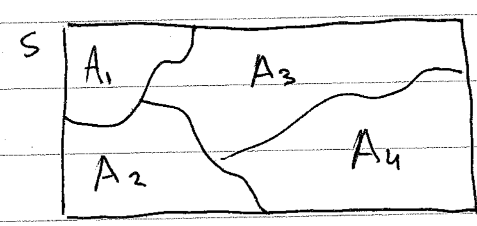
$\left.\begin{array}{l} B \cap C = \emptyset \\ B \cup C = \{3,4,5,6\} \neq S \end{array}\right\} \Rightarrow$ B, C are mutually exclusive, but not collectively exhaustive

$\left.\begin{array}{l} A \cup B \cup C = S \\ A \cap B = \{3\} \neq \emptyset \end{array}\right\} \Rightarrow$ A, B, C are collectively exhaustive, but not mutually exclusive

Let $D = A - B = \{1,2\}$

$\left.\begin{array}{l} B \cup C \cup D = S \\ B \cap C = \emptyset,\ B \cap D = \emptyset,\ C \cap D = \emptyset \end{array}\right\} \Rightarrow$ B, C, D are collectively exhaustive and mutually exclusive.

Def: Sets $A_1, \ldots, A_n$ form a partition of S if $A_1, \ldots, A_n$ are collectively exhaustive and mutually exclusive.



Axiomatic Approach

Consider a random experiment with sample space S, which is the set of all possible outcomes.

<u>Outcomes</u> are elements of S
<u>Events</u> are subsets of S

<u>Note</u>: A random experiment always has an outcome.
Therefore, S is the <u>certain event</u> and
ϕ is the <u>impossible event</u>.

We have the following interpretations for sets in the sample space:

Let A, B be events and $A_1, \ldots, A_n$ a collection of events

$A \cup B$: A or B occurs (possibly both)
$A \cap B$: A and B occur
$\bar{A}$: A does not occur
$A_1, \ldots, A_n$ mutually exclusive : only one A_i may occur (or none)
$A_1, \ldots, A_n$ collectively exhaustive: at least one A_i occurs

As mentioned before, probabilities are assigned to events in this approach.

Any probability assignment must satisfy the following axioms:

(i) $Pr(A) \geq 0$
(ii) $Pr(S) = 1$
(iii) If events A and B are mutually exclusive
$$Pr(A \cup B) = Pr(A) + Pr(B)$$

The axioms provide a consistent framework that we can use to develop several properties and powerful results.

Properties of Probability

1) $Pr(\emptyset) = 0$

Proof: $S = S \cup \emptyset$, $S \cap \emptyset = \emptyset$

By (iii)

$$Pr(S) = Pr(S \cup \emptyset)$$
$$= Pr(S) + Pr(\emptyset)$$
$$\Rightarrow Pr(\emptyset) = 0$$

2) $Pr(A) \leq 1$

<u>Proof:</u> $S = A \cup \bar{A}$, $A \cap \bar{A} = \phi$

$Pr(S) = Pr(A \cup \bar{A})$

$\Rightarrow 1 = Pr(A \cup \bar{A})$, by (ii)

$\Rightarrow 1 = Pr(A) + Pr(\bar{A})$, by (iii)

$\Rightarrow Pr(A) = 1 - Pr(\bar{A})$

$\Rightarrow Pr(A) \leq 1$, $Pr(\bar{A}) \geq 0$ by (i)

3) $Pr(\bar{A}) = 1 - Pr(A)$

<u>Proof:</u> part of proof of 2)

4) $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$

<u>Proof:</u> HW 1

5) If $A \subset B$ then $Pr(A) \leq Pr(B)$

<u>Proof:</u> $B = (B \cap A) \cup (B \cap \bar{A})$

$= A \cup (B \cap \bar{A})$, $A \subset B$

Since $A \cap (B \cap \bar{A}) = \phi$, we have that

$Pr(B) = Pr(A) + Pr(B \cap \bar{A})$

$\geq Pr(A)$, $Pr(B \cap \bar{A}) \geq 0$ by (i)