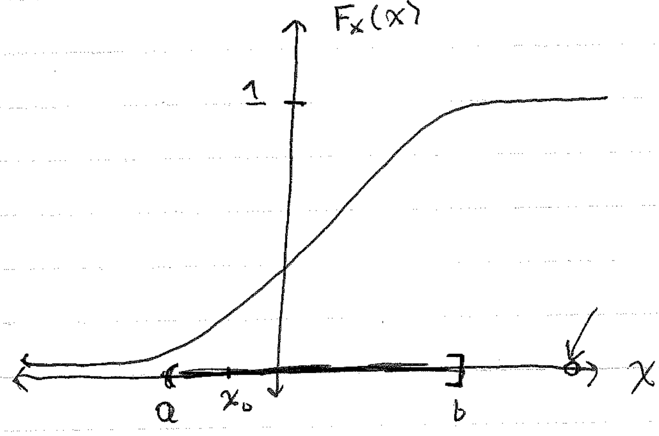
$$p_X(x_i \mid X \in A) = \frac{p_X(x_i)\, I_A(x_i)}{\Pr(X \in A)}$$ indicator function

this is a truncated and scaled version of $p_X(x_i)$

Ex $M = \{X \in (a, b]\}$

$= \{a < X \le b\}$

Want to find $F_X(x \mid M)$, $f_X(x \mid M)$



$\Pr(X \le x_0) = F_X(x_0)$

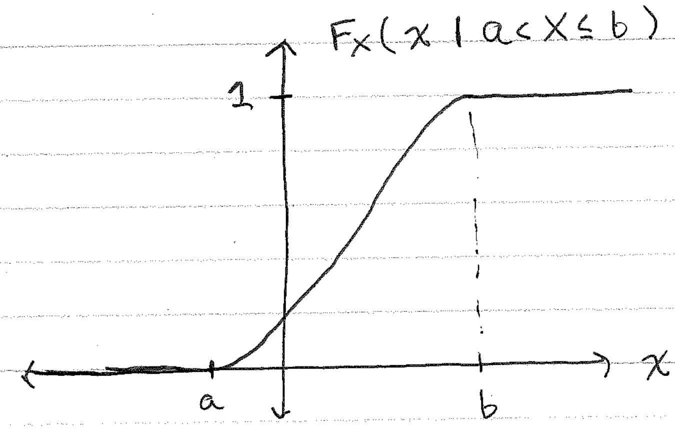
Question. $\Pr(X \le x_0 \mid a < X \le b)$

If $x_0 \le a$

$\Pr(X \le x_0 \mid a < X \le b) = 0$

If $x_0 > b$

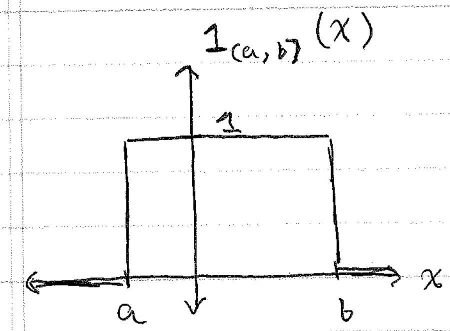
$\Pr(X \le x_0 \mid a < X \le b) = 1$



$$F_X(x \mid a < X \le b) = \frac{\int_{-\infty}^{x} f_X(x')\, \mathbb{1}_{(a,b]}(x')\, dx'}{\Pr(a < X \le b)}$$

Case 1: $x \le a$

$$F_X(x \mid a < X \le b) = \frac{\int_{-\infty}^{x} f_X(x')\, \mathbb{1}_{(a,b]}(x')\, dx}{\Pr(a < X \le b)}$$



$$= \frac{\int_{-\infty}^{x} \cancelto{0}{f_X(x')\, \mathbb{1}}_{(a,b]}(x')\, dx}{\Pr(a < X \le b)}$$

$$= 0$$

Case 2: $x > b$

$$F_X(x \mid a < X \le b) = \frac{\int_{\cancel{-\infty}\, a}^{b} f_X(x')\, \mathbb{1}_{(a,b]}(x')\, dx'}{\Pr(a < X \le b)}$$

$$= \frac{\int_{a}^{b} f_X(x')\, dx'}{\Pr(a < X \le b)} = 1$$

Case 3: $a < x \leq b$

$$F_X(x \mid a < X \leq b) = \frac{\int_{-\infty}^{x} f_X(x')\, \mathbb{1}_{(a,b]}(x')\, dx'}{\Pr(a < X \leq b)}$$

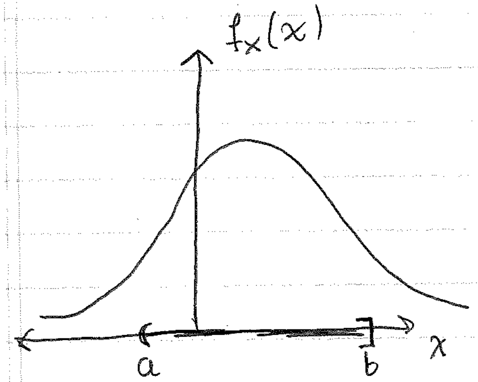
$$= \frac{\int_{a}^{x} f_X(x')\, dx'}{\Pr(a < X \leq b)}$$

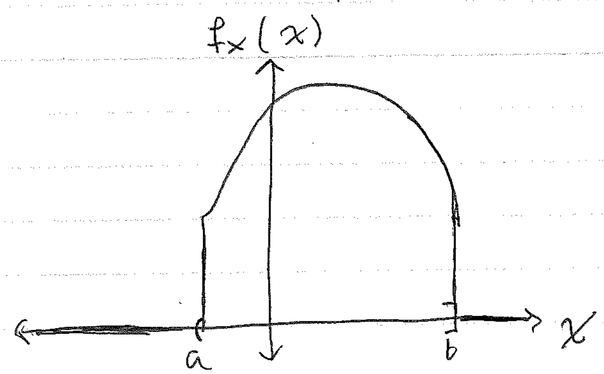
$$= \frac{\Pr(a < X \leq x)}{\Pr(a < X \leq b)}$$

$$= \frac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)}$$

$$F_X(x \mid a < X \leq b) = \begin{cases} 0, & x \leq a \\ \dfrac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)}, & a < x \leq b \\ 1, & x > b \end{cases}$$

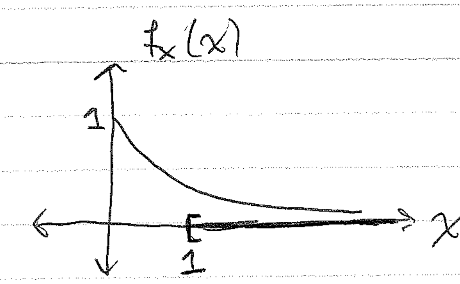
$$f_X(x \mid a < X \leq b) = \begin{cases} \dfrac{f_X(x)}{F_X(b) - F_X(a)}, & a < x \leq b \\ 0, & \text{else} \end{cases}$$





$$f_X(x \mid a < X \le b) = \frac{f_X(x)\ 1_{(a,b]}(x)}{\Pr(a < X \le b)}$$

<u>Ex</u> $f_X(x \mid X \ge 1)$ ← find



$$f_X(x) = e^{-x} u(x)$$

$$F_X(x \mid X \ge 1) = \frac{\int_{-\infty}^{x} f_X(x')\ 1_{[1,\infty)}(x')\, dx'}{\Pr(X \ge 1)}$$

$$\Pr(X \ge 1) = \int_1^{\infty} f_X(x)\, dx$$

$$= \int_1^{\infty} e^{-x}\, dx$$

$$= e^{-1} = 1/e$$

$$F_X \quad \int_{-\infty}^{x} f_X(x')\ 1_{[1,\infty)}(x')\, dx'$$

$$= \int_1^{x} f_X(x')\, dx', \qquad x \ge 1$$

$$= \int_1^{x} e^{-x'}\, dx'$$

$$= e^{-1} - e^{-x}, \qquad x \ge 1$$

$= 0 \quad , \quad x < 1$

$$F_X(x \mid X \geq 1) = \frac{e^{-1} - e^{-x}}{e^{-1}} \quad , \quad x \geq 1$$

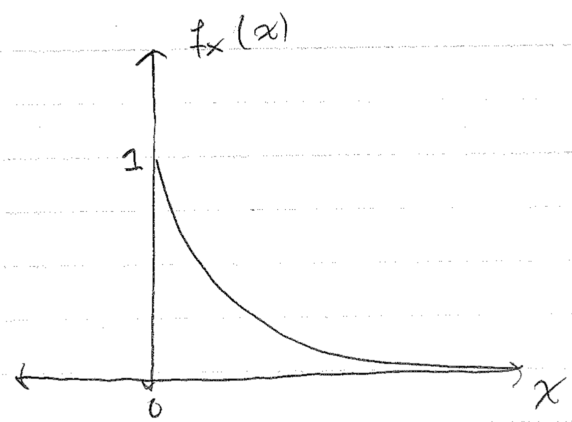
$$= 1 - e^{-(x-1)} \quad , \quad x \geq 1$$

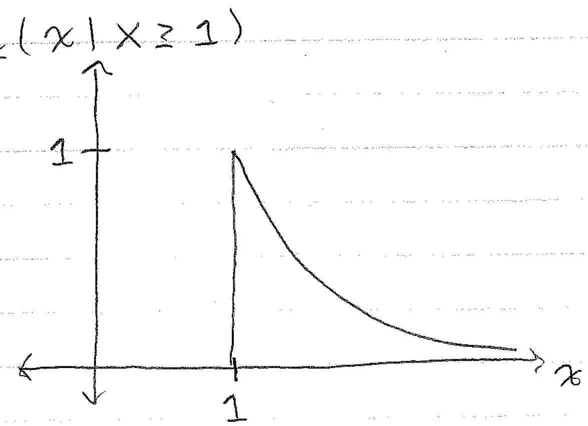
$$= 0 \quad , \quad x < 1$$

$$F_X(x) = 1 - e^{-x} \quad , \quad x \geq 0$$

$$= 0 \quad , \quad 0$$

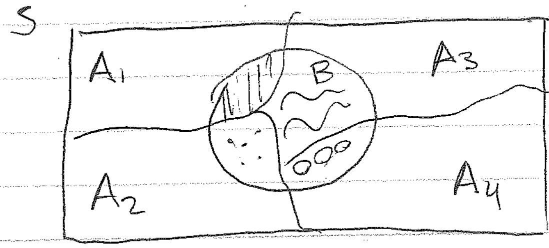
$$f_X(x \mid X \geq 1) = e^{-(x-1)} u(x-1)$$





<u>Memoryless</u>

Total cdf, pdf, pmf, expectation



Let X be r.v., $M_1, \dots, M_n$ be mutually exclusive and collectively exhaustive event.

$$F_X(x) = Pr(X \le x)$$

$$= \sum_{j=1}^{n} Pr(X \le x \mid M_j) Pr(M_j)$$

↑ total prob. thm.

$$= \boxed{\sum_{j=1}^{n} F_X(x \mid M_j) Pr(M_j)}$$

Total cdf

$$f_X(x) = \frac{d}{dx} F_X(x)$$

$$= \frac{d}{dx} \sum_{j=1}^{n} F_X(x \mid M_j) Pr(M_j)$$

$$\boxed{= \sum_{j=1}^{n} f_X(x \mid M_j) Pr(M_j)}$$

Total pdf

$$\boxed{P_X(x_i) = \sum_{j=1}^{n} P_X(x_i \mid M_j) Pr(M_j)}$$

Total pmf

$\cancel{E[g(X)]} =$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} g(x) \left(\sum_{j=1}^{n} f_X(x \mid M_j) \Pr(M_j) \right) dx$$

$$= \sum_{j=1}^{n} \left(\int_{-\infty}^{\infty} g(x) f_X(x \mid M_j)\, dx \right) \Pr(M_j)$$

$$\boxed{= \sum_{j=1}^{n} E[g(X) \mid M_j] \Pr(M_j)}$$

Total Expectation

In particular the mean of X can be found as

$$E[X] = \sum_{j=1}^{n} E[X \mid M_j] \Pr(M_j)$$