ECE 544 **Exam 2 (Three Hour Take Home)** **11/11/13**

Academic Honesty Statement: I am aware of the course policies concerning academic honesty for Professor Krogmeier's section of ECE 544 and for this take home exam. Furthermore, I promise that the work I am submitting with this exam is my own work.

Signature: Solution

Name Printed: ____________________

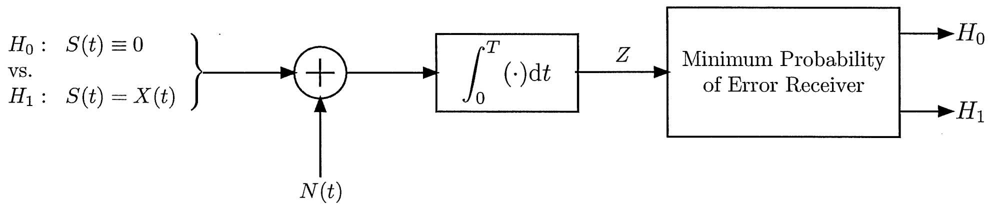
General Instructions:

- **You have 3 hours to complete the exam.**
- Write your name on every page of the exam. Order your answers and number the pages. Do not write on the backs of the pages.
- The exam is open book, notes, calculator, computer, Internet. You may need to use Matlab to solve some of the numerical questions.
- Your work must be explained to receive full credit.
- Point values for each problem are as indicated. The exam totals 100 points.
- The dates and times available to take this exam are:
  - Monday, Nov. 11 from 6 - 9 pm
  - Tuesday, Nov. 12 to Friday, Nov. 15 from 8 am - 5 pm
  - Monday, Nov. 18 from 8 am - 5 pm

  Times are arranged by appointment.

____________________ (Date / Time Start)

____________________ (Date / Time End)



Problem 1. [30 pts. total] A binary communications problem with AWGN $N(t)$ is illustrated above. Under H_0 no signal is transmitted at all. Under H_1 a white Gaussian random process $X(t)$ is transmitted for T seconds. That is, both $N(t)$ and $X(t)$ are zero mean Gaussian random processes with flat power spectral densities of heights $N_0/2$ and $N_1/2$, respectively. We assume that $X(t)$ and $N(t)$ are statistically independent. The waveform part of the receiver is a simple integrator with output Z as indicated. Assume that the two hypotheses are equally likely.

(a) [10 pts.] Using the tools of hypothesis testing find the minimum probability of error detector, with observation Z, as indicated in the block diagram. Simplify and explain.

(b) [10 pts.] Find the minimum average probability of error as a function of the basic problem parameters N_0, N_1, and T. Explain.

(c) [10 pts.] Plot the minimum average probability of error as a function of $\alpha > 0$ where $N_1 = \alpha N_0$ and N_0 corresponds to a noisy resistor of temperature 300 K. Explain the resulting curve.

(a) First find the statistical description of Z under the two hypotheses.

Under H_0

$$Z = \int_0^T N(t)\,dt \quad \longrightarrow \quad Z \text{ is a Gaussian r.v.}$$

$$EZ = 0, \quad Var(Z) = E(Z^2)$$

$$= E\left\{ \int_0^T N(t)\,dt \int_0^T N(s)\,ds \right\}$$

$$= \int_0^T \int_0^T E\{N(t)N(s)\}\,dt\,ds$$

$$= \int_0^T \int_0^T \frac{N_0}{2}\delta(t-s)\,ds\,dt = \frac{N_0}{2}\int_0^T dt = \frac{N_0 T}{2}$$

$$\Longrightarrow Z \sim N(0, N_0T/2) \text{ under } H_0.$$

Under H_1

$$Z = \int_0^T [N(t) + X(t)]\,dt \Longrightarrow$$ a linear comb. of jointly Gaussian rps. is Gaussian.

$$EZ = 0, \quad Var(Z) = E(Z^2)$$

$$= \frac{N_0 + N_1}{2} T$$

Since the two processes are statistically indep.

The min. prob. of error receiver is the Bayes receiver for unif. costs and the test is always a likelihood ratio test:

$$L(z) = \frac{f_1(z)}{f_0(z)} = \frac{\frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-z^2/2\sigma_1^2}}{\frac{1}{\sqrt{2\pi}\,\sigma_0} e^{-z^2/2\sigma_0^2}}$$

$$= \frac{\sigma_0}{\sigma_1} e^{-\left[\frac{1}{2\sigma_1^2} - \frac{1}{2\sigma_0^2}\right] z^2}$$

where $\sigma_0^2 = N_0 T/2 < (N_0 + N_1)T/2 = \sigma_1^2$. The Bayes test compares to the threshold

$$\tau = \frac{\pi_0}{\pi_1} = 1 \quad \text{(in this case).}$$

$$\therefore \quad L(z) = \frac{\sigma_0}{\sigma_1} e^{\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right) z^2/2} \begin{array}{c} \geq \\ < \end{array} 1 \quad \begin{array}{l} \text{say } H_1 \\ \text{say } H_0 \end{array}$$

Simplifying and taking logarithm:

$$\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right) \frac{z^2}{2} \begin{array}{c} \geq \\ < \end{array} \ln\left(\sigma_1/\sigma_0\right) \quad \begin{array}{l} \text{say } H_1 \\ \text{say } H_0 \end{array}$$

Substituting for σ_0, σ_1:

$$\frac{\sigma_0}{\sigma_1} = \left[\frac{N_0 T}{2} \cdot \frac{2}{(N_0 + N_1)T}\right]^{1/2} = \sqrt{\frac{N_0}{N_0 + N_1}}$$

$$\Longrightarrow \ln\left(\sigma_1/\sigma_0\right) = \frac{1}{2} \ln\left(\frac{N_0 + N_1}{N_0}\right)$$

$$\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} = \frac{2}{N_0 T} - \frac{2}{(N_0+N_1)T}$$

$$= \frac{2}{T}\left\{\frac{1}{N_0} - \frac{1}{N_0+N_1}\right\}$$

$$= \frac{2}{T}\,\frac{N_1}{N_0(N_0+N_1)} \Rightarrow \boxed{\left[\frac{1}{2}\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)\right]^{-1} = \frac{T N_0 (N_0+N_1)}{N_1}}$$

Hence a simplified test is:

$$Z^2 \underset{<}{\geq} \frac{1}{2}\ln\left(\frac{N_0+N_1}{N_0}\right)\frac{T N_0 (N_0+N_1)}{N_1} \quad \begin{array}{l}\text{say } H_1\\ \text{say } H_0\end{array}$$

(b) Calculation of error probabilities. Let

$$\eta = \frac{1}{2}\ln\left(\frac{N_0+N_1}{N_0}\right)\frac{T N_0(N_0+N_1)}{N_1}$$

$$P_{e,0} = P_r\left(Z^2 \geq \eta \mid H_0\right)$$

$$P_{e,1} = P_r\left(Z^2 < \eta \mid H_1\right)$$

$$P_e = \frac{1}{2}P_{e,0} + \frac{1}{2}P_{e,1}$$

<u>First compute</u>

$P_{e,1} =$

$$P_r\left(Z^2 < \eta \mid H_1\right) = P_r\left(-\sqrt{\eta} < Z < \sqrt{\eta} \mid H_1\right)$$

$$= P_r\left(\frac{-\sqrt{\eta}}{\sqrt{(N_0+N_1)T/2}} < \frac{Z}{\sqrt{(N_0+N_1)T/2}} \leq \frac{\sqrt{\eta}}{\sqrt{(N_0+N_1)T/2}} \,\Big|\, H_1\right)$$

$$P_{e,1} = \Phi\left(\sqrt{\frac{\eta}{(N_0+N_1)T/2}}\right) - \Phi\left(-\sqrt{\frac{\eta}{(N_0+N_1)T/2}}\right)$$

$$\Phi(-x) = 1 - \Phi(x) \; ; \quad \Phi(x) = 1 - Q(x)$$

$$\Longrightarrow \quad P_{e,1} = 2\,\Phi\left(\sqrt{\frac{\eta}{(N_0+N_1)T/2}}\right) - 1$$

$$= 1 - 2Q\left(\sqrt{\frac{\eta}{(N_0+N_1)T/2}}\right)$$

$$\frac{\eta}{(N_0+N_1)T/2} = \frac{\cancel{\tfrac{1}{2}} \ln\left(\frac{N_0+N_1}{N_0}\right) \cancel{T} N_0 \cancel{(N_0+N_1)}/N_1}{\cancel{\tfrac{1}{2}} \cancel{(N_0+N_1)} \cancel{T}}$$

$$= \frac{N_0}{N_1} \ln\left(\frac{N_0+N_1}{N_0}\right)$$

$$\Longrightarrow$$

$$P_{e,1} = 1 - 2Q\left(\sqrt{\frac{N_0}{N_1} \ln\left(\frac{N_0+N_1}{N_0}\right)}\right)$$

<u>Second compute</u>

$$P_{e,0} = P_r\left(Z^2 \geq \eta \mid H_0\right)$$

$$= P_r\left(Z \geq \sqrt{\eta} \text{ or } Z \leq -\sqrt{\eta} \mid H_0\right)$$

$$= P_r\left(Z \geq \sqrt{\eta} \mid H_0\right) + P_r\left(Z \leq -\sqrt{\eta} \mid H_0\right)$$

$$= 2\,P_r\left(Z \geq \sqrt{\eta} \mid H_0\right)$$

$$P_{e,0} = 2\Pr\left(\frac{Z}{\sqrt{N_0T/2}} \geq \frac{\sqrt{\eta}}{\sqrt{N_0T/2}} \,\middle|\, H_0\right)$$

$$= 2Q\left(\frac{\sqrt{\eta}}{\sqrt{N_0T/2}}\right)$$

$$\frac{\eta}{N_0T/2} = \frac{2\eta}{N_0T} = \frac{\cancel{2}}{\cancel{N_0T}}\frac{1}{\cancel{2}}\ln\left(\frac{N_0+N_1}{N_0}\right)\frac{\cancel{T}\cancel{N_0}(N_0+N_1)}{N_1}$$

$$= \ln\left(\frac{N_0+N_1}{N_0}\right)\frac{N_0+N_1}{N_1}.$$

$$\Longrightarrow P_{e,0} = 2Q\left(\sqrt{\frac{N_0+N_1}{N_1}\ln\left(\frac{N_0+N_1}{N_0}\right)}\right)$$

$$P_e = \frac{1}{2}P_{e,0} + \frac{1}{2}P_{e,1}$$

$$= Q\left(\sqrt{\frac{N_0+N_1}{N_1}\ln\left(\frac{N_0+N_1}{N_0}\right)}\right) - Q\left(\sqrt{\frac{N_0}{N_1}\ln\left(\frac{N_0+N_1}{N_0}\right)}\right)$$

$$+ 1/2$$

(c) $N_1 = \alpha N_0$

$$\Longrightarrow \frac{N_0+N_1}{N_1} = \frac{N_0+\alpha N_0}{\alpha N_0} = \frac{1+\alpha}{\alpha}$$

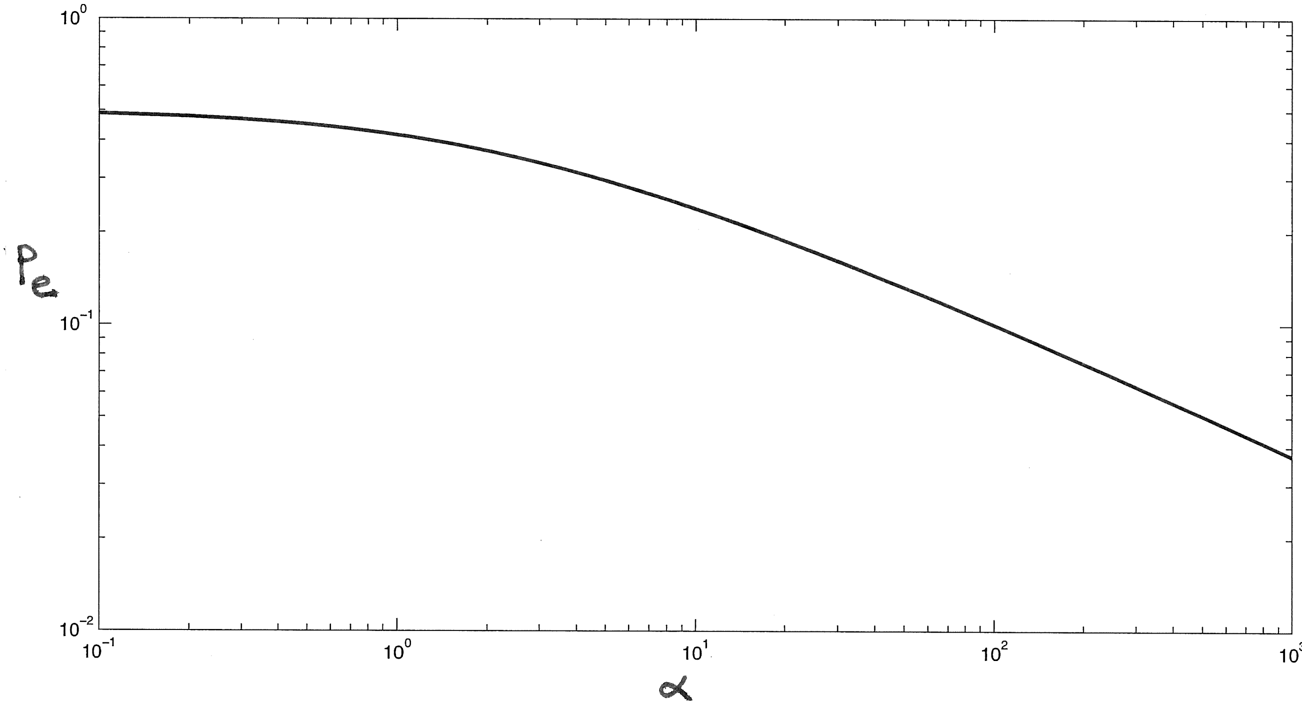
$$\frac{N_0}{N_1} = \frac{N_0}{\alpha N_0} = \frac{1}{\alpha}$$

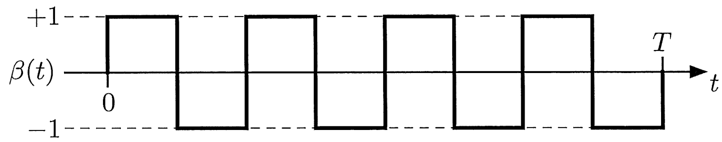
$$\frac{N_0+N_1}{N_0} = \frac{N_0+\alpha N_0}{N_0} = 1+\alpha$$

$$P_e = Q\left(\sqrt{\frac{1+\alpha}{\alpha}\ln(1+\alpha)}\right) - Q\left(\sqrt{\frac{1}{\alpha}\ln(1+\alpha)}\right) + 1/2$$

$$= 1 - \Phi\left(\sqrt{\frac{1+\alpha}{\alpha}\ln(1+\alpha)}\right) - \left\{1 - \Phi\left(\sqrt{\frac{1}{\alpha}\ln(1+\alpha)}\right)\right\} + \frac{1}{2}$$

$$= \frac{1}{2} + \Phi\left(\sqrt{\frac{1}{\alpha}\ln(1+\alpha)}\right) - \Phi\left(\sqrt{\frac{1+\alpha}{\alpha}\ln(1+\alpha)}\right)$$

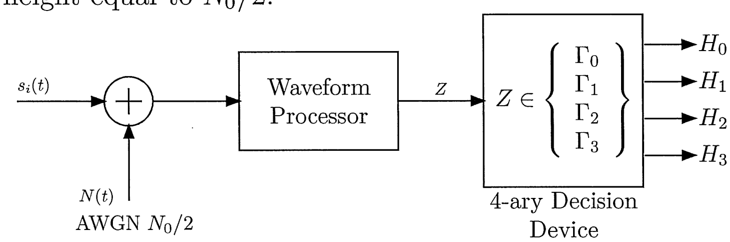




Problem 2. [40 pts. total] Consider the 4-ASK system with transmitted signals

$$s_i(t) = u_i\sqrt{2}A\beta(t)\cos(2\pi f_c t + \phi)$$

for $0 \le t \le T$ and $i = 0, 1, 2, 3$. The pulse $\beta(t)$ is defined as shown above, $f_c \gg 1/T$, and $u_0 = -3$, $u_1 = -1$, $u_2 = +1$, and $u_3 = +3$. The signals are received in AWGN of power spectral density height equal to $N_0/2$.



The optimal receiver has the structure of a waveform processor followed by a 4-ary decision device as shown above.

(a) [5 pts.] What is the form of the information lossless waveform processor? Give reasons though you do not need to derive the result from first principles.

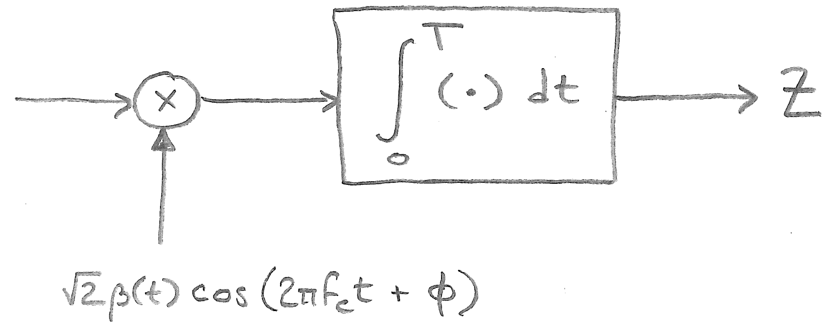
(b) [8 pts.] For the proper choice in (a) find the statistical models of the random variable Z for each of the four hypotheses.

(c) [10 pts.] Assuming equally likely signals, find the minimum average probability of error decision device (i.e., specify the decision regions), the value of the minimum average probability of error, and plot it versus SNR. Be clear on your choice of SNR parameter.

(d) [12 pts.] Find the minimax receiver for this problem (i.e., specify the decision regions), the value of the resulting average probability of error, and plot it versus SNR. Be clear on your choice of SNR parameter.

(e) [5 pts.] Compare the two receivers from (c) and (d).

ⓐ Waveform Processor



ⓑ Noise part of Z: $\quad N = \int_0^T N(t)\sqrt{2}\beta(t)\cos(2\pi f_c t + \phi)\, dt$

$\Longrightarrow$ N is a zero mean Gaussian rv. Its variance can be computed from

$$Var(N) = E(N^2)$$

$$= \int_0^T \frac{N_0}{2} \cdot 2\beta^2(t)\cos^2(2\pi f_c t + \phi)\, dt$$

$$= N_0 \int_0^T \cos^2(2\pi f_c t + \phi)\, dt \qquad \text{since } \beta^2(t) = \begin{cases} 1 & 0 \le t \le T \\ 0 & \text{else} \end{cases}$$

$$= \frac{N_0}{2} T$$

Signal part of Z:

$$S_i(t)\sqrt{2}\beta(t)\cos(2\pi f_c t + \phi)$$

$$= u_i \sqrt{2} A \beta(t)\cos(2\pi f_c t + \phi)\sqrt{2}\beta(t)\cos(2\pi f_c t + \phi)$$

$$= u_i A \beta^2(t)\, 2\cos^2(2\pi f_c t + \phi)$$

Therefore using $\beta^2(t) = 1$ and trig identity and $f_c >> 1/T$

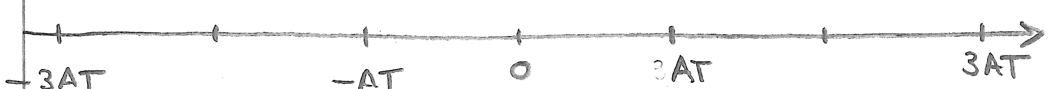
$$S_i = \int_0^T s_i(t)\sqrt{2}\,\beta(t)\cos(2\pi f_c t + \phi)\,dt$$

$$= u_i A T$$

In summary the statistical model is :

$$H_0 : \quad Z \sim N(-3AT, N_oT/2)$$
$$H_1 : \quad Z \sim N(-AT, N_oT/2)$$
$$H_2 : \quad Z \sim N(+AT, N_oT/2)$$
$$H_3 : \quad Z \sim N(+3AT, N_oT/2)$$

(c) As shown in class the receiver is the minimum distance receiver



From minimum distance

$$\Gamma_0 = \{ z : -\infty < z \le -2AT \}$$
$$\Gamma_1 = \{ z : -2AT < z \le 0 \}$$
$$\Gamma_2 = \{ z : 0 < z \le 2AT \}$$
$$\Gamma_3 = \{ z : 2AT < z < \infty \}$$

$$P_{c,0} = P_{c,3} = P_r\left(2AT < Z \mid H_3\right)$$

$$= P_r\left(\frac{2AT - 3AT}{\sqrt{N_0T/2}} < \frac{Z - 3AT}{\sqrt{N_0T/2}} \,\middle|\, H_3\right)$$

$$= Q\left(\frac{-AT}{\sqrt{N_0T/2}}\right) = Q\left(-\sqrt{\frac{2A^2T}{N_0}}\right)$$

$$P_{e,0} = P_{e,3} = 1 - Q\left(-\sqrt{\frac{2A^2T}{N_0}}\right)$$

$$= Q\left(\sqrt{\frac{2A^2T}{N_0}}\right)$$

$$P_{c,1} = P_{c,2} = P_r\left(0 < Z \leq 2AT \mid H_2\right)$$

$$= P_r\left(\frac{-AT}{\sqrt{N_0T/2}} < \frac{Z - AT}{\sqrt{N_0T/2}} \leq \frac{2AT - AT}{\sqrt{N_0T/2}} \,\middle|\, H_2\right)$$

$$= \Phi\left(\frac{AT}{\sqrt{N_0T/2}}\right) - \Phi\left(\frac{-AT}{\sqrt{N_0T/2}}\right)$$

$$= \Phi\left(\sqrt{\frac{2A^2T}{N_0}}\right) - \Phi\left(-\sqrt{\frac{2A^2T}{N_0}}\right)$$

$$= 1 - Q\left(\sqrt{\frac{2A^2T}{N_0}}\right) - Q\left(\sqrt{\frac{2A^2T}{N_0}}\right)$$

$$= 1 - 2Q\left(\sqrt{\frac{2A^2T}{N_0}}\right)$$

$$\Longrightarrow P_{e,1} = P_{e,2} = 2Q\left(\sqrt{\frac{2A^2T}{N_0}}\right)$$

Then the average prob. of error will be

$$P_e = \frac{1}{4}P_{e,0} + \frac{1}{4}P_{e,1} + \frac{1}{4}P_{e,2} + \frac{1}{4}P_{e,3}$$

$$= \frac{1}{2}P_{e,0} + \frac{1}{2}P_{e,1}$$

$$= \frac{1}{2}Q\left(\sqrt{\frac{2A^2T}{N_0}}\right) + Q\left(\sqrt{\frac{2A^2T}{N_0}}\right)$$

$$= \frac{3}{2}Q\left(\sqrt{\frac{2A^2T}{N_0}}\right)$$

(d) The minimax receiver should choose the decision regions $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3$ st.

$$\max\{P_{e,0}, P_{e,1}, P_{e,2}, P_{e,3}\}$$

is minimized. It turns out that an equalizer rule is the solution and from symmetry can see that the minimax decision regions will be

$$\Gamma_0 = \{z : -\infty < z \le -\eta\}$$

$$\Gamma_1 = \{z : -\eta < z \le 0\}$$

$$\Gamma_2 = \{z : 0 < z \le \eta\}$$

$$\Gamma_3 = \{z : \eta < z < \infty\}$$

where solve for η st.

$$\Pr(0 < Z \le \eta \mid H_2) = \Pr(\eta < Z < \infty \mid H_3)$$

$$\Pr(0 < Z \le \eta \mid H_2)$$

$$= \Pr\left(\frac{0 - AT}{\sqrt{N_0 T/2}} < \frac{Z - AT}{\sqrt{N_0 T/2}} \le \frac{\eta - AT}{\sqrt{N_0 T/2}} \,\middle|\, H_2 \right)$$

$$= \Phi\left(\frac{\eta - AT}{\sqrt{N_0 T/2}} \right) - \Phi\left(\frac{-AT}{\sqrt{N_0 T/2}} \right)$$

$$\Pr(\eta < Z < \infty \mid H_3)$$

$$= \Pr\left(\frac{\eta - 3AT}{\sqrt{N_0 T/2}} < \frac{Z - 3AT}{\sqrt{N_0 T/2}} \,\middle|\, H_3 \right)$$

$$= Q\left(\frac{\eta - 3AT}{\sqrt{N_0 T/2}} \right) = 1 - \Phi\left(\frac{\eta - 3AT}{\sqrt{N_0 T/2}} \right)$$

$\Longrightarrow$ Solve for η s.t.

$$\Phi\left(\frac{\eta - AT}{\sqrt{N_0 T/2}} \right) - 1 + \Phi\left(\sqrt{\frac{2A^2T}{N_0}} \right) = 1 - \Phi\left(\frac{\eta - 3AT}{\sqrt{N_0 T/2}} \right)$$

We will need to solve the previous for the equalizer threshold η and it will depend on the SNR. Hence need to renormalize so the SNR parameter is evident.

Let $\rho = \dfrac{2A^2T}{N_0}$ which incidently is the SNR param. in the perf. equation of part c to which we hope to compare.

$\eta' \triangleq \eta/AT$

$$\frac{\eta - AT}{\sqrt{N_0T/2}} = \sqrt{\frac{2(\eta - AT)^2}{N_0T}} = \sqrt{\frac{2A^2T^2(\eta/AT - 1)^2}{N_0T}} = \sqrt{\frac{2A^2T}{N_0}(\eta'-1)^2}$$

$$= \sqrt{\rho(\eta'-1)^2} \qquad (\text{assuming } \eta > AT)$$

$$\frac{\eta - 3AT}{\sqrt{N_0T/2}} = -\sqrt{\frac{2(\eta - 3AT)^2}{N_0T}} = -\sqrt{\frac{2A^2T^2}{N_0T}(\eta'-3)^2}$$

$$= -\sqrt{\rho(\eta'-3)^2} \qquad (\text{assuming } \eta < 3AT)$$

Then if don't want to worry about the range of η' can just take the terms $(\eta'-1)$ and $(\eta'-3)$ outside of the radicals and account there for the signs. Then the η' equation is

$$\Phi\left((\eta'-1)\sqrt{\rho}\right) - 1 + \Phi(\sqrt{\rho}) = 1 - \Phi\left((\eta'-3)\sqrt{\rho}\right)$$

which should yield

$\eta'(\rho)$

Before writing the code to solve for η' it makes sense to figure out the range of the SNR param. ρ and for that lets plot the error prob. of part ©.

$$P_{e,\text{ part}©} = \frac{3}{2} Q\left(\sqrt{\frac{2A^2T}{N_0}}\right)$$

Lets rewrite the SNR in terms of the average energy per bit, which we get by dividing the average energy per symbol by the number of bits per symbol.

$$\|s_i\|^2 = \int_0^T s_i^2(t)\,dt = u_i^2 \cdot 2 \cdot A^2 \int_0^T \beta^2(t) \cos^2(2\pi f_c t + \phi)\,dt$$

$$= 2u_i^2 A^2 T/2 = u_i^2 A^2 T$$

$$\|s_0\|^2 = 9A^2T = \|s_3\|^2; \quad \|s_1\|^2 = A^2T = \|s_2\|^2$$

$$E_s = \frac{1}{4} 9A^2T + \frac{1}{4} A^2T + \frac{1}{4} A^2T + \frac{1}{4} 9A^2T$$

$$= \frac{A^2T}{4}\left[9+1+1+9\right] = \frac{20}{4} A^2T = 5A^2T$$

$$E_b = E_s/2 = \frac{5}{2} A^2T$$

The new SNR parameter will be

$$\tilde{\rho} = E_b/N_0 = \frac{5}{2}\frac{A^2T}{N_0} = \frac{5}{4}\rho$$

$$\Rightarrow \rho = \frac{4}{5}\tilde{\rho}$$

$$P_{e,\text{ part ⓒ}} = \frac{3}{2} Q\left(\sqrt{\tfrac{4}{5}\tilde{\rho}}\right) = \frac{3}{2}\left[1 - \Phi\left(\sqrt{\tfrac{4}{5}\tilde{\rho}}\right)\right]$$

$$\tilde{\rho}_{dB} = 10 \log_{10} \tilde{\rho}$$

A range of interesting values of $\tilde{\rho}_{dB}$ is then found to be 0 to 15 dB.

Pseudo Code:

for a range of values of $\tilde{\rho}_{dB} = 10 \log_{10} \tilde{\rho}$

Solve

(*) $$\Phi\left((\eta'-1)\sqrt{\tfrac{4}{5}\tilde{\rho}}\right) + \Phi\left((\eta'-3)\sqrt{\tfrac{4}{5}\tilde{\rho}}\right) + \Phi\left(\sqrt{\tfrac{4}{5}\tilde{\rho}}\right) - 2 = 0$$

for $\eta'(\tilde{\rho})$

Substitute the value of $\eta'(\tilde{\rho})$ into

$$P_{e,3} = \Phi\left((\eta'-3)\sqrt{\tfrac{4}{5}\tilde{\rho}}\right) = Q\left((3-\eta')\sqrt{\tfrac{4}{5}\tilde{\rho}}\right)$$

and use the fact that all the conditional error probs are equal to conclude

$$P_{e,\text{ part ⓓ}} = P_{e,3}$$

end

Plot $P_{e,\text{ part ⓓ}}$ and compare to $P_{e,\text{ part ⓒ}}$.

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%% Problem 2 of Exam 2


trho_dB = 0:0.1:15;                    %Range of SNR parameter
rho_tilde
trho = 10 .^ (trho_dB / 10);

% Symbol error probability for the minimum probability of error
receiver
Pe_part_c = 1.5*(1 - normcdf(sqrt(0.8*trho)));

figure(1)
hold off
semilogy(trho_dB,Pe_part_c,'r')
grid
xlabel('SNR parameter E_b/N_0 in dB')
ylabel('Probability of a Symbol Error')

% Setup for calculation of the minimax threshold for part (d)
etaprime = ones(1,length(trho));    %Initializing

for k = 1:length(trho)
    f = @(x)evaluate_probs(x,trho(k));       %Used to send SNR
parameter
    etaprime(k) = fzero(f,2);                %Solve for the
threshold
end

figure(2)
plot(trho_dB,etaprime)
grid
title('Plot the minimax threshold parameter etaprime vs. SNR')
xlabel('SNR parameter E_b/N_0 in dB')
ylabel('Threshold')

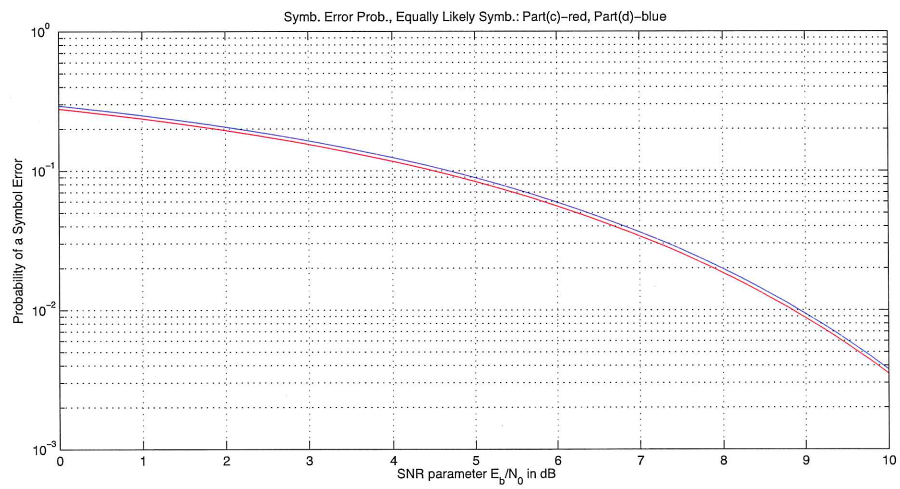
figure(1)

hold on
Pe_part_d = normcdf((etaprime - 3) .* sqrt(0.8*trho));
semilogy(trho_dB,Pe_part_d,'b')
title('Symb. Error Prob., Equally Likely Symb.: Part(c)-red, Part
(d)-blue')
xlabel('SNR parameter E_b/N_0 in dB')
ylabel('Probability of a Symbol Error')
```

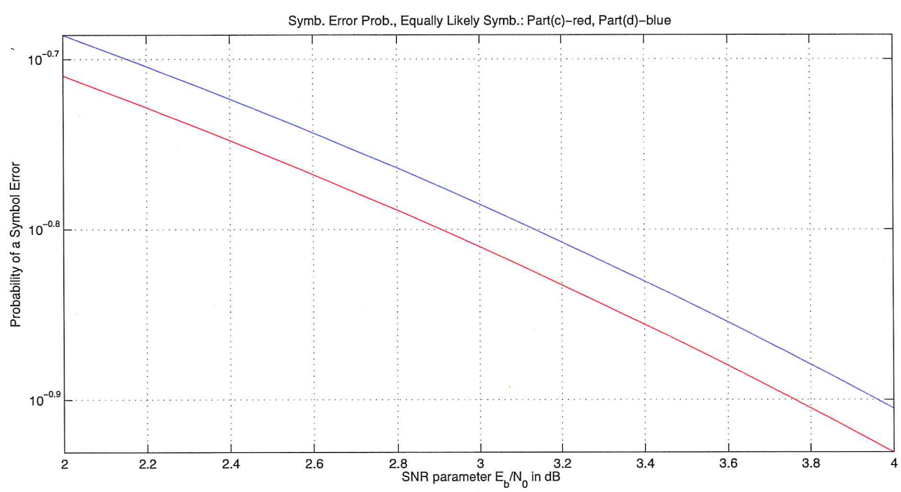
```
function  f = evaluate_probs(x,y)
   f = normcdf((x-1)*sqrt(0.8*y)) + normcdf((x-3)*sqrt(0.8*y)) +
...
      normcdf(sqrt(0.8*y)) - 2;
```

Function evaluates * from p. 8.

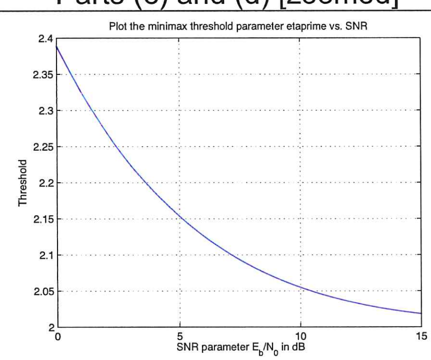
Exam2 Problem 2 Figures



Parts (c) and (d)



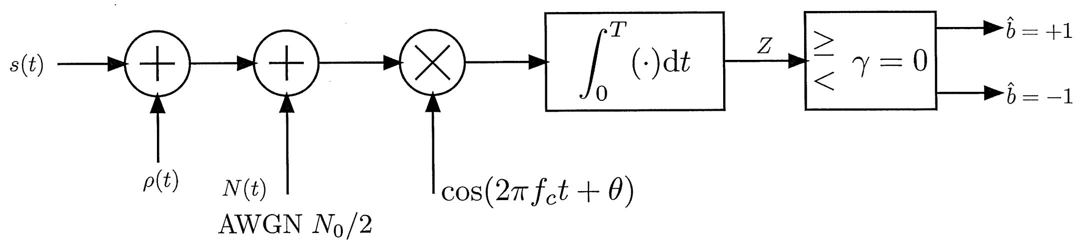
Parts (c) and (d) [zoomed]



Minimax Threshold vs. SNR

(e) Note (1) $P_{e,\text{part(c)}} < P_{e,\text{part(d)}}$ as expected.

(2) As SNR gets large the minimax and min prob receivers are approx. the same.



Problem 3. [30 pts. total] The block diagram above represents a BPSK transmission system where there are two transmission paths from transmitter to receiver. The signal received on the direct path is

$$s(t) = Abp_T(t)\cos(2\pi f_c t + \theta)$$

where $b = +1$ or -1 represents the (equally likely) binary data and $f_c \gg 1/T$. The signal received on the secondary path is

$$\rho(t) = RAbp_T(t)\cos(2\pi f_c t + \theta + \Phi)$$

where Φ is uniform on $[0,\ 2\pi)$, R is a Rayleigh random variable[1], and Φ and R are independent of each other and of the noise $N(t)$.

(a) [5 pts.] Find an expression for $P_e(r, \phi)$ the conditional error probability given $R = r$ and $\Phi = \phi$.

(b) [10 pts.] Give an integral expression for the unconditional error probabilty $P_e = \mathrm{E}\{P_e(R, \Phi)\}$.

(c) [15 pts.] Evaluate the expression to simplify to something expressible in terms of $Q(\cdot)$.

[1]The Rayleigh probability density function is

$$f_R(r) = \begin{cases} (r/\lambda^2)\exp(-r^2/2\lambda^2) & r \geq 0 \\ 0 & r < 0 \end{cases}$$

where λ is a parameter.

(a) Conditioned on $R=r$ and $\Phi=\varphi$ the received signal is

$$s(t)+\rho(t) = A b\, p_T(t) \cos(2\pi f_c t + \theta) + r A b\, p_T(t) \cos(2\pi f_c t + \theta + \varphi)$$

$$= A b\, p_T(t) \left[\cos(2\pi f_c t + \theta) + r \cos(2\pi f_c t + \theta + \varphi) \right]$$

$b = \pm 1$ equally likely.

The signal part of Z (conditioned on b, $R=r$, $\Phi=\varphi$) will be denoted $S=s$

$$s = \int_0^T [s(t)+\rho(t)] \cos(2\pi f_c t + \theta)\, dt$$

$$= Ab \int_0^T \cos^2(2\pi f_c t + \theta)\, dt + Abr \int_0^T \cos(2\pi f_c t + \theta) \cos(2\pi f_c t + \theta + \varphi)\, dt$$

$$= \frac{Ab}{2} T + \frac{Abr}{2} \cos\varphi\, T$$ under the standard assumptions

$$= \frac{AbT}{2} \left[1 + r\cos\varphi \right]$$

The noise part of Z will be denoted by N. As in prev. derivations it is Gaussian and of zero mean. Its variance is

$$E(N^2) = \frac{N_0}{4} T$$

So we have reduced to a standard problem:

Conditioned on
$b=+1, R=r, \Phi=\varphi$:

$$Z \sim N\left(\frac{AT}{2}[1+r\cos\varphi], \frac{N_0T}{4}\right)$$

Conditioned on
$b=-1, R=r, \Phi=\varphi$:

$$Z \sim N\left(-\frac{AT}{2}[1+r\cos\varphi], \frac{N_0T}{4}\right)$$

In the problem statement we are given that $\gamma=0$, which makes sense because the priors are equal and the two means above are always symm wrt $Z=0$.

The decision rule is:

$$N+\frac{ATb}{2}\left[1+r\cos\phi\right] \begin{array}{c} > \\ < \end{array} \tau=0 \quad \begin{array}{l} \text{say } \hat{b}=+1 \\ \text{say } \hat{b}=-1 \end{array}$$

Let $P_{e,-1}(r,\phi)$ and $P_{e,+1}(r,\phi)$ denote the indicated conditional error probabilities. Then we can show

$$P_e(r,\phi)=\frac{1}{2}P_{e,-1}(r,\phi)+\frac{1}{2}P_{e,+1}(r,\phi)$$

<u>Hypothesis : b = +1</u>

$$P_{e,+1}(r,\phi) = P_r\left(N + \frac{AT}{2}\left[1 + r\cos\phi\right] < 0\right)$$

$$= P_r\left(\frac{N}{\sqrt{N_0T/4}} < \frac{-\frac{AT}{2}(1 + r\cos\phi)}{\sqrt{N_0T/4}}\right)$$

$$= \Phi\left(-(1 + r\cos\phi)\sqrt{\frac{A^2T}{N_0}}\right)$$

$$= Q\left((1 + r\cos\phi)\sqrt{\frac{A^2T}{N_0}}\right)$$

<u>Hypothesis : b = -1</u>

$$P_{e,-1}(r,\phi) = P_r\left(N - \frac{AT}{2}(1 + r\cos\phi) > 0\right)$$

$$= P_r\left(\frac{N}{\sqrt{N_0T/4}} > \frac{\frac{AT}{2}(1 + r\cos\phi)}{\sqrt{N_0T/4}}\right)$$

$$= Q\left((1 + r\cos\phi)\sqrt{\frac{A^2T}{N_0}}\right)$$

$$\therefore\ P_e(r,\phi) = Q\left((1 + r\cos\phi)\sqrt{\frac{A^2T}{N_0}}\right)$$

(b) The unconditional error probability would be

$$P_e = E\left\{ P_e(R, \Phi) \right\}$$

$$= \int_0^\infty \int_0^{2\pi} \underbrace{P_e(r,\phi)} \frac{1}{2\pi} \frac{r}{\lambda^2} e^{-r^2/2\lambda^2} \, d\phi \, dr$$

$$\int_0^\infty \int_0^{2\pi} \int_{(1+r\cos\phi)\sqrt{A^2T/N_0}}^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{2\pi} \frac{r}{\lambda^2} e^{-r^2/2\lambda^2} \, dx \, d\phi \, dr$$

(c) Evaluate the above. Seems very nasty. A change of variables seems to be in order. But there is a <u>much</u> simpler way.

<u>Fact</u> R, Φ stat. indep with $R \sim$ Rayleigh with param λ and Φ unif on $[0, 2\pi)$ then

$$X \triangleq R\cos\Phi, \quad Y = R\sin\Phi$$

are stat. indep, 0 mean Gaussian rvs with variance λ^2.

Then going back to the receiver see that the decision statistic

$$Z = \frac{AbT}{2}\left[1 + X\right] + N$$

where $b = \pm 1$ depending on the hypothesis, $X \sim N(0, \lambda^2)$, $N \sim N(0, N_0T/4)$ and $X \perp\!\!\!\perp N$.

<u>Under $b = +1$</u> we make error iff

$$\frac{AT}{2}(1 + X) + N < 0$$

and <u>under $b = -1$</u> we make error iff

$$-\frac{AT}{2}(1 + X) + N > 0$$

$$\Longleftrightarrow$$

$$\frac{AT}{2}(1 + X) - N < 0$$

Therefore, since N and $-N$ have the same distribution these probabilities must be equal.

$\therefore P_{e, b=+1}$

$$= P_r\left(\frac{AT}{2} + \frac{AT}{2}X + N < 0\right)$$

$$= P_r\left(\frac{AT}{2}X + N < -\frac{AT}{2}\right)$$

Note that the r.v. $\frac{AT}{2}X + N$ is zero mean Gaussian with variance

$$\frac{A^2T^2}{4}\lambda^2 + \frac{N_oT}{4}$$

$$\therefore P_{e,b=+1} = P_e$$

$$= P_r\left(\frac{\frac{AT}{2}X+N}{\sqrt{\frac{A^2T^2}{4}\lambda^2+\frac{N_oT}{4}}} < \frac{-AT/2}{\sqrt{\frac{A^2T^2}{4}\lambda^2+\frac{N_oT}{4}}}\right)$$

$$= Q\left(\frac{AT/2}{\sqrt{\frac{A^2T^2}{4}\lambda^2+\frac{N_oT}{4}}}\right)$$

$$= Q\left(\frac{1}{\sqrt{\lambda^2 + N_o/A^2T}}\right)$$

<u>Check</u>: What does it mean if $\lambda \to 0$ and how does this expression behave? Does the limit make sense?