(1.) Let $\varepsilon > 0$ be given. $\exists\ N \ni \forall\ |x| > N,\quad |f(x)| < \varepsilon/2$

$(*)\quad \Rightarrow \forall\ x, y \in \mathbb{R} \ni |x| > N\ \&\ |y| > N,\quad |f(x) - f(y)| \leq |f(x)| + |f(y)| < \varepsilon$

$(**)\ \left\{ \right.$ Next, since f is continuous on $[-N-1, N+1]^n$ (compact), $\exists\ \delta > 0 \ni \forall\ x, y \in [-N-1, N+1]^n\ \&\ |x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$.

WLOG $\delta < 1/2$ (by choosing δ possibly smaller).

Now we show f is unif. cont. on $\mathbb{R}$.
Let δ be as above, & $x, y \in \mathbb{R} \ni |x - y| < \delta$.

If $|x|\ \&\ |y| > N$ we have $|f(x) - f(y)| < \varepsilon$ by $(*)$

Suppose $|x| \leq N$ since $|x - y| < \delta < 1/2$, $\Rightarrow |y| < N+1$.
$\Rightarrow |f(x) - f(y)| < \varepsilon$ by $(**)$. $\boxtimes$

$\left([-N, N]^n = \underbrace{[-N, N] \times \cdots \times [-N, N]}_{n}\right)$.

② $f:\mathbb{R}\to\mathbb{R}$ continuous & 1-1.

Pf: Suppose (by contradiction) f is not strictly increasing, nor strictly decreasing. $\Rightarrow \exists\ x_1 < x_2 < x_3 \ni f(x_1) \le f(x_2)\ \ f(x_2) \ge f(x_3)$

or $f(x_1) \ge f(x_2)\ \ f(x_2) \le f(x_3)$.

WLOG. $f(x_1) \le f(x_2)$ & $f(x_2) \ge f(x_3)$.

Since f is 1-1, $f(x_1) < f(x_2)$ & $f(x_2) > f(x_3)$, & $f(x_1) \ne f(x_3)$.

WLOG $f(x_1) > f(x_3)$.

Set $g(x) = f(x) - f(x_1)$. $\Rightarrow g(x)$ is also continuous & 1-1 on $\mathbb{R}$.

But $g(x_1) = 0$, $g(x_2) > 0$ & $g(x_3) < 0 \Rightarrow \exists\ c \ni x_2 < c < x_3$

$\ni g(c) = 0$ (by IVT). ※ Contradiction.

$\Rightarrow f$ is strictly increasing or decreasing. ▨

(3) Let $s_n, t_n > 0 \ni \quad 0 < \underline{\lim} \frac{s_n}{t_n} \leq \overline{\lim} \frac{s_n}{t_n} < \infty$.

Let $\varepsilon > 0$. $\quad \exists\ N \ni \forall\ n \geq N, \ \frac{s_n}{t_n} > \underline{\lim} \frac{s_n}{t_n} - \varepsilon$.

Since ε arbitrary, we have $\frac{s_n}{t_n} > \left(\underline{\lim} \frac{s_n}{t_n}\right)/2$.

Further, let $\alpha = \min\limits_{1,\dots,N} \left\{\frac{s_n}{t_n}\right\} > 0$ & choose $M \in \mathbb{N}$, M large enough so that $\frac{1}{M} < \alpha$ & $\frac{1}{M} < \underline{\lim} \left(\frac{s_n}{t_n}\right)/2$.

$\Rightarrow \forall n \ \frac{s_n}{t_n} > 1/M$ ✓

Next, $\exists\ N$ (different N) $\ni \forall\ n \geq N$, $\frac{s_n}{t_n} < \overline{\lim} \frac{s_n}{t_n} + \varepsilon < \infty$.

So, let $\beta = \max\limits_{1,\dots,N} \left\{\frac{s_n}{t_n}\right\}$ & choosing M larger we have

$M > \overline{\lim} \left\{\frac{s_n}{t_n}\right\} + \varepsilon$ & $M > \beta$.

$\Rightarrow \forall n \quad \frac{1}{M} < \frac{s_n}{t_n} < M$ as desired. $\boxtimes$

4. (a) $a_n = (2+(-1)^n)^n z^n$

using root test we require

$$1 > \overline{\lim_n} |a_n|^{1/n} = \overline{\lim_n} |z||2+(-1)^n| = 3|z|$$

So $|z| < \frac{1}{3}$ $\boxed{R = 1/3}$

(b) $a_n = \dfrac{(3n)!\, n!}{(4n)!} x^n$.

using root test we require

$$1 > \overline{\lim_n} \left| \frac{(3n+3)!\,(n+1)!\,(4n)!\,x^{n+1}}{(3n)!\,n!\,(4n+4)!\,x^n} \right| = |x| \overline{\lim_n} \left| \frac{(3n+3)(3n+2)(3n+1)\cdot(n+1)}{(4n+4)(4n+3)(4n+2)(4n+1)} \right|$$

$$= |x| \cdot \frac{3^3}{4^4}$$

$\boxed{\text{ie } |x| < \frac{4^4}{3^3} = R}$

(c) $a_n = x^{n^n}$. using the root test we require

$\overline{\lim_n} |x|^n < 1 \iff \boxed{|x| < 1 = R}$

(d) $a_n = n^2 x^n$ using the ratio test we require

$$1 > \overline{\lim_n} \left| \frac{(n+1)^2 x^{n+1}}{n^2 x^n} \right| = |x| \overline{\lim_n} \left| \left(\frac{n+1}{n}\right)^2 \right| = |x|$$

$\boxed{R = 1}$

☑

5. Let Σa_n converge absolutely, & $\{b_n\}$ a bounded sequence.

(a) Show $\Sigma a_n b_n$ converges.

(b) What if Σa_n just converges?

(a) It suffices to show $\Sigma a_n b_n$ converges absolutely.

$\exists\, M < \infty \ni |b_n| < M \ \forall n.$

$\Rightarrow \Sigma |a_n b_n| = \Sigma |a_n||b_n| \leq M \Sigma |a_n| < \infty$ since $\Sigma |a_n|$ converges. ✓

(b) Let $a_n = \frac{(-1)^n}{n} \Rightarrow \Sigma a_n$ converges, but not absolutely.

Let $b_n = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ even.} \end{cases}$ Clearly b_n is bounded, & $\Sigma a_n b_n = \Sigma |a_n| = \infty$.

$\boxtimes$

#6 Let f be continuous on $[a,b]$ & differentiable on (a,b), with $f(a)=f(b)=0$. $\forall\, \lambda \in \mathbb{R}\ \exists\ c \in (a,b) \ni f'(c) = \lambda f(c)$.

WARNING!: f' is not assumed to be continuous. You cannot use IVT on $\frac{f'(x)}{f(x)}$.

Pf: Let $\lambda \in \mathbb{R}$ be given.

If $f \equiv 0$, $\Rightarrow f' \equiv 0$ & the result is trivial.

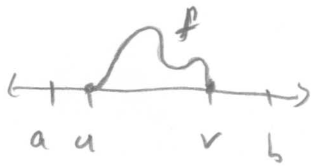
WLOG $\exists\, x_0 \ni f(x_0) \neq 0$.

If $f(x_0) < 0$, replace f & λ by $-f$, $-\lambda$ respectively, &

WLOG $f(x_0) > 0$.

Set $u = \sup\{x < x_0 : f(x) = 0\}$, $v = \inf\{x > x_0 : f(x) = 0\}$ $\Rightarrow a \le u < x_0 < v \le b$ & $f(u) = 0 = f(v)$ & $\forall\, x \in (u,v)\ f(x) > 0$

(i.e.



)

Now we want to find $c \ni \frac{f'(c)}{f(c)} = \lambda$, so we notice that $g(x) = \log(f(x))$ is differentiable on (u,v) ($f > 0$ here) & $g' = \frac{f'}{f}$.

$\lim_{x \to u^+} g(x) = -\infty$ since $\lim_{x \to u^+} f(x) = 0$ & log is continuous.

Let $x_1 \in (u,v)$. $\exists\, x_2 \in (u, x_1) \ni g(x_2) < g(x_1) - 1$ (Since $g(x) \to -\infty$ as $x \downarrow u$.)
Illy $\exists\, x_3 \in (u, x_2) \ni g(x_3) < g(x_2) - 1$, &
inductively, we find $x_1 > x_2 > \cdots > x_n > \cdots > u \ni g(x_{n+1}) < g(x_n) - 1\ \forall n \in \mathbb{N}$

6. (cont'd) Now $\{x_n\}$ is decreasing & bdd below, hence Cauchy.

Choose $M > |\lambda|$.

$\exists N \ni \forall n \geq m \geq N \quad |x_m - x_n| = x_m - x_n < \frac{1}{M}$.

$$\Rightarrow \quad \frac{g(x_N) - g(x_{N+1})}{x_N - x_{N+1}} > M \qquad \left(\begin{array}{l} g(x_N) - g(x_{N+1}) > 1 \\ \& \ x_N - x_{N+1} < 1/M \end{array} \right.$$

By the MVT $\exists \ c_1 \in (x_{N+1}, x_N) \ni \dfrac{g(x_N) - g(x_{N+1})}{x_N - x_{N+1}} = g'(c_1)$

i.e. $g'(c_1) > |\lambda|$.

Next, using $\lim\limits_{x \to v^-} g(x) = -\infty$, an almost identical proof

shows $\exists \ c_2 \in (u, v) \ni g'(c_2) < -|\lambda|$.

Now apply Theorem 5.12, i.e. since $g'(c_2) < \lambda < g'(c_1)$

$\exists \ c \in (c_1, c_2) \ni g'(c) = \lambda$.

i.e. $\dfrac{f'(c)}{f(c)} = \lambda$. $\boxtimes$

7) Let f be uniformly continuous on $\mathbb{R}$

Set $\varepsilon = 1$ $\exists \delta > 0 \ni |f(x) - f(y)| < \varepsilon \quad \forall |x-y| < \delta$.

Let $x \in \mathbb{R}$, $x > 0$. $\exists\ n_x = n \ni n(\frac{\delta}{2}) \leq x < (n+1)\frac{\delta}{2}$

$|f(x)| \leq |f(x) - f(x - \delta/2)| + |f(x - \delta/2) - f(x - 2(\delta/2))| + \cdots$

$+ |f(x - n(\frac{\delta}{2})) - f(0)| + |f(0)|$

But $|f(x - k\frac{\delta}{2}) - f(x - (k+1)\frac{\delta}{2})| < \varepsilon = 1$ by $(*)$

$\Rightarrow |f(x)| < n\varepsilon + |f(0)| = n + |f(0)|$

But, from $(*)$, $n \leq \frac{2x}{\delta}$ so $|f(x)| < \frac{2x}{\delta} + |f(0)|$

Choose $A = |f(0)|$, $B = \frac{2}{\delta}$

The same proof for $x < 0$ holds, but we obtain $n \leq \frac{2|x|}{\delta}$.

$\Rightarrow |f| \leq B|x| + A$ as desired. $\boxtimes$

(8.) "$\Rightarrow$" Let $f:[a,b]\to\mathbb{R}$ be continuous & monotone.

WLOG $f\nearrow$. (If not, replace f w/ $-f$.)

$\forall\ x\in(a,b)$ $\quad f(a)\le f(x)\le f(b)$ $\quad\Rightarrow f([a,b])\subset[f(a),f(b)]$

Further, by IVT $\forall\ y\in(f(a),f(b))\ \exists\ a<x<b \ni f(x)=y$.

$\Rightarrow [f(a),f(b)]\subset f([a,b])$. ✓

"$\Leftarrow$" Let $f:[a,b]\to\mathbb{R}$ w/ f monotone & $f([a,b])$ an interval.

WLOG $f\nearrow$.

$\Rightarrow f$ only has simple discontinuities.

Let $c\in[a,b]$ be such a discontinuity (by contradiction).

$$(*)\left\{\begin{array}{ll}\forall\ a\le x<c, & f(x)\le f(c),\\ \forall\ c<x\le b, & f(c)\le f(x),\end{array}\right\}\Rightarrow \lim_{x\to c^-}f(x)\le f(c)\le\lim_{x\to c^+}f(x).$$

Since f is discontinuous at c, one of the above inequalities is strict.

If $\lim_{x\to c^-}f(x)<f(c)$, we have by $(*)$ that

$f([a,b])\subset(-\infty,\alpha)\,\dot\cup\,[f(c),\infty)$ where $\alpha=\lim_{x\to c^-}f(x)$.

※ Contradiction.

The other case is similar. ☒