

1. Below you are given an input, $x(t)$, and the impulse response of a system, $h(t)$. Let $y(t) = S\{x(t)\} = x(t) * h(t)$ be the output of the system.

Some formulas you may need are:

- Fourier series analysis

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

- Fourier series synthesis

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

- Frequency response given the impulse response

$$H(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

~~should be~~ $\int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$

Show all of your work for full credit.

- (a) (2 points) Find the Fourier series coefficients, a_k , of

$$x(t) = 1 + 2e^{j2\pi t} + 3e^{j\frac{5\pi}{3}t}$$

Use the $T_0 = 6$ for the period.

- (b) (3 points) Find the frequency response, $H(\omega)$, of

$$h(t) = e^{-2t} u(t)$$

- (c) (3 points) Find the output Fourier series coefficients, b_k , of $y(t)$.

- (d) (2 points) Write an expression for the output, $y(t)$. The final answer should not contain a summation operator.

a) Using the synthesis equation

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{6}t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{\pi}{3}t} \\ &= 1 + 2e^{j2\pi t} + 3e^{j\frac{5\pi}{3}t} \end{aligned}$$

By comparing $x(t)$ with the synthesis equation:

$$a_0 = 1, a_6 = 2, a_5 = 3$$

all others must be zero

$$\begin{aligned}
 b) \quad h(t) &= e^{-2t} u(t) \\
 H(\omega) &= \int_0^\infty e^{-2t} e^{j\omega t} dt = \int e^{-t(z+j\omega)} dt \\
 &= -\frac{1}{z+j\omega} [e^{-t(z+j\omega)}]_0^\infty = -\frac{1}{z+j\omega} [e^{\int_0^\infty -\omega(j\omega) dt} - e^0] \\
 &= \boxed{\frac{1}{z+j\omega} = H(\omega)}
 \end{aligned}$$

$$c) \quad y(t) \xrightarrow{\text{FS}} b_k = H(R\omega_0) e^{jk\omega_0 t}$$

$$b_0 = H(0) \cdot 1 = \frac{1}{2} \cdot 1 = \frac{1}{2}$$

$$b_6 = H(6 \cdot \frac{\pi}{3}) \cdot 2 = \frac{1}{2+j2\pi} \cdot 2 = \frac{1}{1+j\pi}$$

$$b_5 = H(5 \cdot \frac{\pi}{3}) \cdot 3 = \frac{1}{2+j\frac{5\pi}{3}} \cdot 3 = \frac{9}{6+j5\pi}$$

$$\begin{aligned}
 d) \quad y(t) &= \sum_{k=-\infty}^{\infty} b_k e^{jk\frac{\pi}{3}t} \\
 &= \frac{1}{2} + \frac{1}{1+j\pi} e^{j\pi t} + \frac{9}{6+j5\pi} e^{j\frac{5\pi}{3}t}
 \end{aligned}$$

a) Direct integration approach:

This method is somewhat trickier and longer than using the synthesis equation. I recommend using the synthesis equation when you have complex exponentials.

Looking at just one of them, $x_1(t) = 3e^{j\frac{\pi}{3}t} \xrightarrow{FS} c_R$

$$c_R = \frac{1}{6} \int_{-3}^3 3e^{j\frac{\pi}{3}t} e^{-jk\frac{\pi}{3}t} dt = \frac{3}{6} \int_{-3}^3 e^{-jt\frac{\pi}{3}(k-5)} dt$$

$$= \frac{3}{6} \cdot \frac{-1}{j\frac{\pi}{3}(k-5)} \left[e^{-jt\frac{\pi}{3}(k-5)} \right]_{-3}^3$$

$$= \frac{1}{2} \cdot \frac{-1}{j\frac{\pi}{3}(k-5)} \left[e^{-j\pi(k-5)} - e^{j\pi(k-5)} \right]$$

$$= \frac{1}{\frac{\pi}{3}(k-5)} \sin(\pi(k-5))$$

Notice that $\sin(\pi(k-5)) = 0$ for all integer k . However, the denominator equals zero if $k=5$. To find c_5 , we can either apply L'Hopital's rule or go back a few steps and plug in $k=5^-$:

$$\begin{aligned} c_5 &= \frac{1}{2} \int_{-3}^3 e^{-jt\frac{\pi}{3}(k-5)} dt \Big|_{k=5^-} = \frac{1}{2} \int_{-3}^3 e^{-jt\frac{\pi}{3}(0)} dt \\ &= \frac{1}{2} \int_{-3}^3 dt = \frac{1}{2} \cdot 6 = 3 \end{aligned}$$

$$\Rightarrow c_5 = 3, \quad c_k = 0 \text{ for } k \neq 5^-$$

While it's good to know this method, it is almost too tedious to perform under the time pressure of an exam.