MA544 Qual Review Problems 7 **Bridges** **Summer 08**

Dingdongs of Glory

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1. Let $(X, \mathcal{F}, \mu)$ be a probability space and suppose $f \in L^1(\mu)$. Prove that
$$\lim_{p\to 0} \|f\|_p = \exp\left(\int_X \log|f| d\mu\right).$$
Hint: $\forall x > 0, -\log(x)$ is convex, $\log(x) \le x - 1$, and $\phi(p) = \dfrac{x^p - 1}{p}$ is monotone increasing in p.

2. Let $(X, \mathcal{F}, \mu)$ be a probability space. Suppose $f \in L^\infty(\mu)$ and $\|f\|_\infty > 0$.

   (a) Prove that
   $$\lim_{n\to\infty} \|f\|_n = \|f\|_\infty.$$
   (b) Prove that
   $$\lim_{n\to\infty} \frac{\int_X |f|^{n+1} d\mu}{\int_X |f|^n d\mu} = \|f\|_\infty.$$
   (c) Are the results above true for any finite measure space? for any measure space?

3. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $1 < p < \infty$. Assume that $f : X \to \mathbb{R}$ is $\mathcal{M}$-measurable and satisfies:
$$\mu(\{x : |f(x)| > y\}) \le \frac{c_0}{y^p}$$
where c_0 is independent of $y > 0$. Let $1 \le r < p$. Show that
$$\int_X |f|^r d\mu \le c\mu(X)^{1-r/p}$$
where c depends only on c_0, r, p.

4. Let $f \in C([0, 1])$. Show that there is a sequence of odd polynomials $\{p_n(x)\}$ with $p_n \to f$ uniformly on $[0, 1]$ if and only if $f(0) = 0$.

5. Suppose that $f_n(x)$ is a sequence of functions in $AC[0,1]$ which are increasing (in x) and for which $f_n(0) = 0$ for all n. Let

$$g(x) = \sum_1^\infty f_n(x).$$

Prove that if $g(1) < \infty$, then $g \in AC[0,1]$.

6. Let $(X, \mathcal{F}, \mu)$ be a finite measure space and let $1 < p < \infty$. Suppose f_n is a sequence of measurable functions in $L^p(\mu)$ with $\|f_n\|_p \leq 1$ for all n and $f_n \to f$ a.e. Prove that

$$\int_X f_n g \; d\mu \to \int_X fg \; d\mu$$

for all $g \in L^q(\mu)$, where q is the conjugate index of p.

7. Discuss the convergence of the sequence

$$\left\{ \int_0^1 x^{1/n} |f(x)| dx \right\}_{n=1}^\infty$$

for $f \in L^1([0,1])$.

8. Let $(X, \mathcal{A}, \mu)$ be a finite measure space. If f is $\mathcal{A}$-measurable, let

$$E_n = \{x \in X : n - 1 \leq |f(x)| < n\}.$$

Show that $f \in L^1$ if and only if $\sum_{n=1}^\infty n\mu(E_n) < \infty$.