※ Review: Sampling Theory and DTFT.

1) Sampling a sinewave $x_a(t) = e^{j\omega_a t}$

$$x[n] = x_a(nT_s) = e^{j\omega_a n T_s} = e^{j\omega_d n}$$

where $\omega_d = \omega_a T_s = \frac{\omega_a}{F_s}$

ω_d: DT freq

ω_a: CT freq.

Now: $e^{j\omega_a t} \xleftrightarrow{\text{CTFT}} X_a(\omega) = 2\pi\,\delta(\omega - \omega_a)$

Thus: $X_s(\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} 2\pi\,\delta\left(\omega - \omega_a - k\frac{2\pi}{T_s}\right)$

$X_s(\omega)$: sampled signal

(sidenote) " $\delta(ax) = \frac{1}{|a|}\delta(x)$ "

$\int_{-\infty}^{\infty} \delta(ax)\,dx$ use change of variables : $u = ax$

$du = a\,dx$

$= \int_{-\infty}^{\infty} \delta(u)\frac{1}{a}\,du = \frac{1}{a}\underbrace{\int_{-\infty}^{\infty}\delta(u)\,du}_{=1} = \frac{1}{a}$: area under $\delta(ax)$

(assume $a > 0$)

So we can conclude $\delta(ax) = \frac{1}{a}\delta(x)$ since both $\delta(x)$ and $\delta(ax)$ are only nonzero at $x=0$.

Thus, $X_s(\omega) = \frac{1}{T_s} \sum_k 2\pi\, \delta\left(\frac{1}{T_s}(\omega T_s - \omega_a T_s - k 2\pi)\right)$

Using the property of delta func. above,

$$= \cancel{\frac{1}{T_s}} \sum_k 2\pi\, \cancel{T_s}\, \delta(\omega T_s - \omega_a T_s - k 2\pi)$$

Finally $X(\omega) = X_s(F_s \omega) = X_s\left(\frac{\omega}{T_s}\right)$

$$= \sum_k 2\pi\, \delta(\omega - \omega_d - k 2\pi)$$

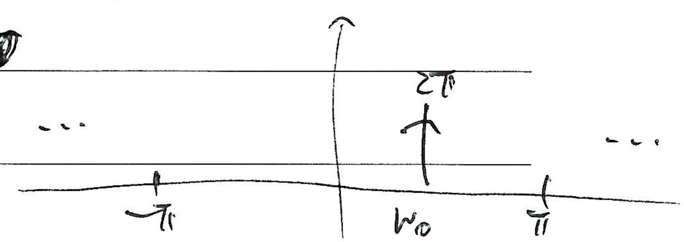
where $\omega_d = \omega_a T_s = \frac{\omega_a}{F_s}$

* comment : There is no amplitude scaling by the sampling rate.

It verifies the FT pair $e^{j\omega_0 n} \xleftrightarrow{DTFT} \sum_k 2\pi\, \delta(\omega - \omega_0 - k 2\pi)$

* Dirac delta func. at ω_0 and every integer multiple of 2π away from ω_0

$\Rightarrow$ DTFT is periodic with period 2π.

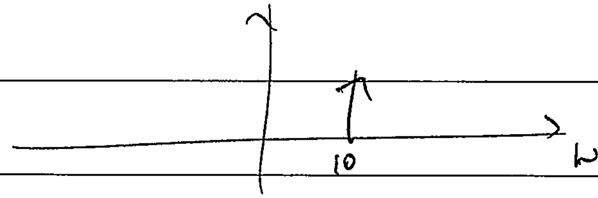


for $-\pi < \omega < \pi$

(sidenote) max. freq. for a sinewave with freq w_0 is w_0

→ Nyquist rate is twice the freq of the sinewave.

e.g. sinewave : 10 Hz



2) Definition of DTFT

- It is motivated by passing (infinite duration) sinewave thru LTI DT system

$$x[n] = e^{jw_0 n} \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] \\ &= \sum_{k} h[k]\, e^{jw_0(n-k)} \\ &= e^{jw_0 n} \underbrace{\sum_{k} h[k]\, e^{-jw_0 k}}_{= H(w_0)} \end{aligned}$$

where $H(w) = \sum_{k} h[k]\, e^{-jkw}$

DTFT : $x[n] \longleftrightarrow X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

3) Two ways to provide $X(\omega)$ from $X_a(\omega)$

① First, compress $X_a(\omega)$ by the sampling rate, F_s, then repeat / replicate result obtained every integer multiple of 2π.

②. First, replicate $X_a(\omega)$ every integer multiple of $\frac{2\pi}{T_s}$ to create $X_s(\omega)$. then compress analog freq axis by F_s. ($\omega_d = \frac{\omega_a}{F_s}$) "F_s"

- The 2nd method is preferred since it takes into account aliasing first, but either way is fine.
- Either way, there is also an amplitude scaling by the sampling rate F_s.

1. $$x_s(t) = x_a(t)\left(\sum_n \delta(t-nT_s)\right)$$

$$= \sum_n \underbrace{x_a(nT_s)}_{x[n]} \delta(t-nT_s)$$

$$X_s(\omega) = \text{CTFT}\{x_s(t)\}$$

$$= \sum_n x[n]\, e^{-j\omega n T_s} = \sum_n x[n]\, e^{-j(\underset{\text{CT freq}}{\omega} T_s) n}$$

$$X_d(\omega) = X_s(\omega)\Big|_{\underset{\omega_d}{\omega} = \underset{\omega_a}{\omega} T_s} = \sum_n x[n]\, e^{-j\underset{\text{DT freq}}{\omega} n}$$

"$\omega_d = \omega_a T_s$"

2. since $$\sum_n \delta(t-nT_s) \overset{\text{CTFT}}{\longleftrightarrow} \frac{2\pi}{T_s}\sum_k \delta\left(\omega - k\frac{2\pi}{T_s}\right)$$

and since $$x_a(t)\sum_n \delta(t-nT_s) \underset{\text{CTFT}}{\longleftrightarrow} \frac{1}{2\pi}\mathcal{F}\{x_a(t)\} * \mathcal{F}\left\{\sum_n \delta(t-nT_s)\right\}$$

$$= X_s(\omega)$$

We obtain the classic result that sampling in time domain gives rise to replications of the spectrum (CTFT) of $x_a(t)$ at every integer multiple of $\frac{2\pi}{T_s}$

$$X_s(w) = \frac{1}{\cancel{2\pi}} X_a(w) * \frac{\cancel{2\pi}}{T_s} \sum_k \delta\left(w - k\frac{2\pi}{T_s}\right)$$

$$= \frac{1}{T_s} \sum_k X_a\left(w - k\frac{2\pi}{T_s}\right)$$

$$= \frac{1}{T_s} \sum_k X_a\left(\frac{1}{T_s}(w T_s - k2\pi)\right)$$

CT freq

since $w_d = w T_s$

w_a (CT freq)

$$x[n] = x_a(t)\big|_{t=nT_s} \xleftrightarrow[\text{DTFT}]{} X(w) = \frac{1}{T_s} \sum_k X_a\left(\frac{1}{T_s}(w - k2\pi)\right)$$

DT freq.

<u>Example</u> $x_a(t) \xrightarrow{A/D} x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$

- $x[n] = x_a(nT_s)$ where $x_a(t) = T_s \frac{\pi}{10}\left(\frac{\sin(10t)}{\pi t}\right)^2$

  and $T_s = \frac{2\pi}{30}$ ($w_s = 30$)

- $h[n] = 2\frac{\sin\left(\frac{\pi}{6}n\right)}{\pi n}\cos\left(\frac{5\pi}{6}n\right)$

a) Check aliasing b) Plot $|X(w)|$ c) Plot $|Y(w)|$ d) $\sum_{n=-\infty}^{\infty} y^2[n]$