MA598 - Complex Analysis Qual Prep - Summer 2014
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Supplement to PS3 - Review of relevant concepts

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Theorem 1 (Cauchy's Integral Theorem) *Let Ω be a simply connected open subset of $\mathbb{C}$ and suppose $\gamma : [0,1] \to \Omega$ is a rectifiable curve in Ω. Then for any $f \in \mathcal{O}(\Omega)$,*

$$\int_\gamma f(z)\, dz = 0.$$

Exercise Suppose that $\Omega \subset \mathbb{C}$ has compact closure, with $\partial\Omega$ smooth. Show that if a holomorphic map f extends continuously to $\partial\Omega$, then the conclusion remains valid for all rectifiable curves in $\overline{\Omega}$.

Theorem 2 (Morera's Theorem) *Suppose Ω is a domain, and $f : \Omega \to \mathbb{C}$ a continuous function. Then $f \in \mathcal{O}(\Omega)$ if*

$$\int_\gamma f(z)\, dz = 0$$

for all triangles γ in Ω.

Theorem 3 (Cauchy's Integral Formula) *Let $\Omega \subset \mathbb{C}$ be a domain and $f \in \mathcal{O}(\Omega)$. Suppose for $z_0 \in \Omega$ and $r > 0$ that $\overline{B_r(z_0)} \subset \Omega$. Then*

$$f(z_0) = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(z)}{z - z_0}\, dz,$$

where the integral is computed using the positive orientation.

Theorem 4 (Residue Theorem) *Suppose Ω is a simply connected domain, $a_1, \cdots, a_n \in \Omega$, and $f \in \mathcal{O}(\Omega - \{a_1, \cdots, a_n\})$. Let γ be a positively oriented simple closed curve which encloses a_i, $1 \le i \le n$. Then*

$$\int_\gamma f(z)\, dz = \sum_{i=1}^n \operatorname{Res}(f, a_i),$$

where $\operatorname{Res}(f, a_i)$ denotes the residue of f at a_i.

Remark For those who have taken Algebraic Topology, think about this result from the perspective of the de Rham cohomology of the punctured plane.