EE 301 Exam 1 2/11/13

Name: Solution

General Instructions:

- **You have 60 minutes to complete the exam.**
- Write your name on every page of the exam.
- The exam is closed book and closed notes. Calculators are not allowed.
- Your work must be explained to receive full credit. All plots must be carefully drawn with axes labeled.
- Point values for each problem are as indicated. The exam totals 100 points.
- If you finish the exam during the first 50 minutes, you may turn it in and leave. During the last 10 minutes you must remain seated until we pick up exams from everyone.
- Problems labeled with **LO** indicate that the problem is used to determine student satisfaction of course learning objectives.

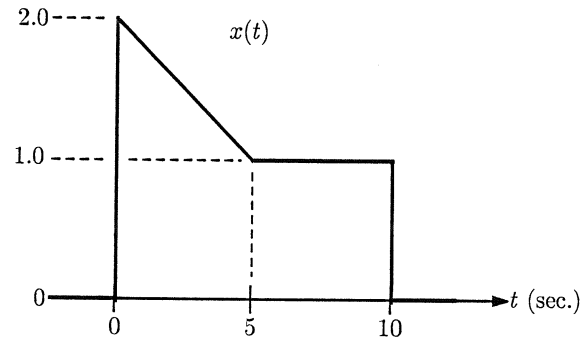
This exam is for Krogmeier's section of 301.

Do not open the exam until you are told to begin.

Name: Solution

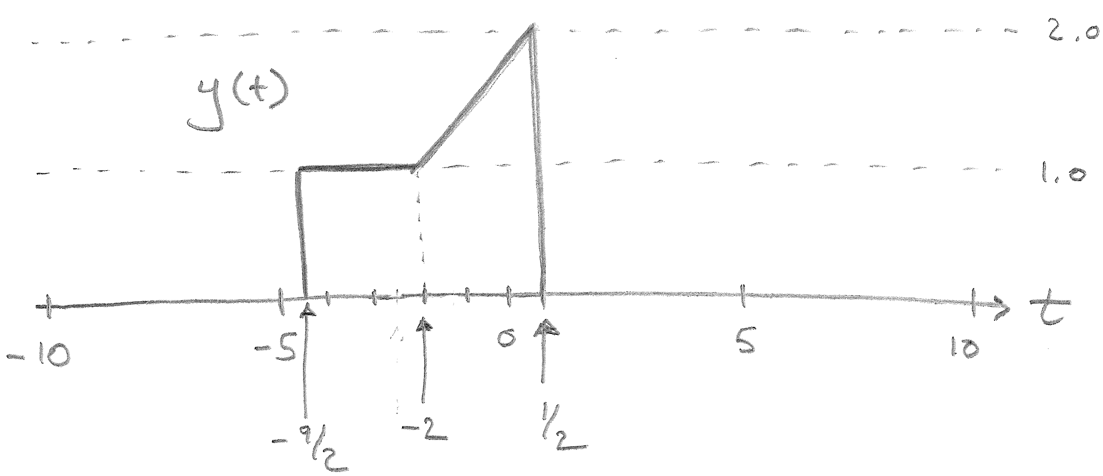
Problem 1. *Signal Properties.* [30 pts. total, **LO-i**]
This problem has three unrelated parts.

(a) [5 pts.] If $x(t)$ is the signal shown, plot $y(t) = x(1-2t)$.



$y(t)$ will be some sort of stretched/compressed and time reversed version of $x(t)$. Solve for values of t st

$$1 - 2t = 0 \longrightarrow t = 1/2$$
$$= 5 \longrightarrow -2t = 4 \longrightarrow t = -2$$
$$= 10 \longrightarrow -2t = 9 \longrightarrow t = -9/2$$



Problem 1. (cont'd.) Name: Solution

(b) [10 pts.] For each of the signals below simplify to the maximum possible extent and find the average power in each.

(b-1) $x_1(t) = 2\cos(20\pi t) - 2\sin(20\pi t)$. Try to write in form $A\cos(20\pi t+\theta)$

$$= 2\,\text{Re}\left\{ e^{j20\pi t} + j e^{j20\pi t} \right\}$$

$$= 2\,\text{Re}\left\{ (1+j) e^{j20\pi t} \right\} = 2\,\text{Re}\left\{ \sqrt{2}\, e^{j\pi/4} e^{j20\pi t} \right\}$$

$$= 2\sqrt{2}\cos(20\pi t + \pi/4)$$

$\therefore$ From 201 average power formula for a sinusoid

$$P = \frac{1}{2}(2\sqrt{2})^2 = 4$$

(b-2) $x_2(t) = 2\cos(20\pi t + \pi/4) - 2\sin(20\pi t + 3\pi/4)$.

$$= 2\,\text{Re}\left\{ e^{j(20\pi t + \pi/4)} + j e^{j(20\pi t + 3\pi/4)} \right\}$$

$$= 2\,\text{Re}\left\{ \left(e^{j\pi/4} + j e^{j3\pi/4} \right) e^{j20\pi t} \right\}$$

$$= 2\,\text{Re}\left\{ \underbrace{\left(1 + e^{j\pi/2} e^{j\pi/2} \right)}_{1 + (j)^2 = 1-1 = 0} e^{j\pi/4} e^{j20\pi t} \right\}$$

$$\therefore x_2(t) = 0 \implies P = 0$$

Problem 1. (cont'd.) Name: **Solution**

(c) [15 pts.] Let $w(t) = e^{-3t}u(t)$ and define

$$z(t) = \sum_{k=-\infty}^{\infty} w(t-10k).$$

(To simplify calculations be sure to make reasonable engineering approximations. Explain.)

(c-1) Find the energy of $w(t)$.

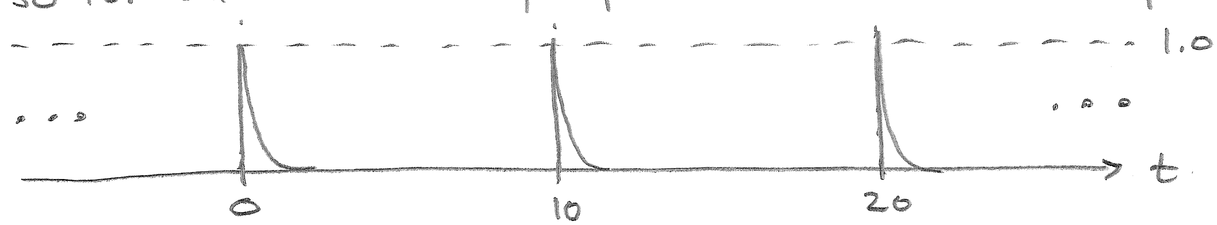
$$E = \int_0^\infty w^2(t)\,dt = \int_0^\infty e^{-6t}\,dt = -\frac{1}{6}e^{-6t}\Big|_{t=0}^{\infty}$$

$$= 1/6$$

(c-2) Is $z(t)$ periodic or non-periodic? If periodic, find its fundamental period. Make a rough plot using reasonable assumptions. Explain.

$z(t)$ is clearly periodic of period $T = 10$ because of the repeating shifts in the sum. Note that $w(10) = e^{-30} \approx 0$ so for all intents and purposes there is no overlap:

$z(t)$



(c-3) Find the average power of $z(t)$.

$$P = \frac{\text{energy of } w}{\text{period}} = \frac{1/6}{10} = \frac{1}{60}$$

Name: **Solution**

Problem 2. *System Properties.* [40 pts. total, **LO-i**]
This problem has four parts. For each system shown, choose the correct statements. You must give some explanation of your choices in order to receive full credit.

$$x[n] \longrightarrow \boxed{y[n] = \sin(2\pi(n-1)/3)\cdot x[n]} \longrightarrow y[n]$$

(a) [10 pts.] The system above is (circle all correct choices, and explain):

(a-1) Memoryless *or* with memory. Is memoryless since output at a specific time n only depends on input at the same time n. $\sin(2\pi(n-1)/3)$ is just a time-varying gain.

(a-2) Causal *or* non-causal. Memoryless is always causal.

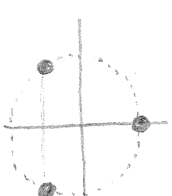
(a-3) BIBO stable *or* unstable. Suppose there exists a $B<\infty$ st. $|x[n]|\le B$ for all $n\in\mathbb{Z}$. Then

$$|y[n]| = |\sin(2\pi(n-1)/3)|\cdot|x[n]| \le |x[n]| \le B \quad \forall n.$$

because $|\text{sine wave}|\le 1$

$$\Longrightarrow |y[n]|\le B \quad \forall n.$$

(a-4) Time invariant *or* time varying.



Pretty obvious since gain is time varying. Let $G_n \triangleq \sin(2\pi(n-1)/3)$

$\Longrightarrow G_1 = 0,\ G_2 = \sin 2\pi/3 = \sqrt{3}/2,\ G_3 = \sin 4\pi/3 = -\sqrt{3}/2,\ G_4 = 0 \cdots$ ie periodic

(a-5) Linear *or* non-linear.

Say $y_1[n] = G_n x_1[n]$
$y_2[n] = G_n x_2[n]$

Then

$$G_n(\alpha x_1[n] + x_2[n])$$
$$= \alpha G_n x_1[n] + G_n x_2[n]$$
$$= \alpha y_1[n] + y_2[n]$$

Let $x_1[n] = \begin{cases} 1 & \text{if } n = 4k+1 \ (\text{a mult of } 4 \text{ plus } 1) \\ 0 & \text{else} \end{cases}$

$$x_1[n] \longrightarrow \boxed{\text{above syst.}} \longrightarrow y_1[n] = 0 \ \forall n$$

$$x_1[n-1] = x_2[n] \longrightarrow \boxed{\text{above syst.}} \longrightarrow y_2[n] = \begin{cases} \sqrt{3}/2 & \text{if } n = 4k+2,\ k\in\mathbb{Z} \\ 0 & \text{else} \end{cases}$$

But $y_2[n] \ne y_1[n-1]$

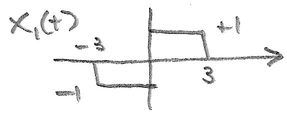
Problem 2. (cont'd.) Name: Solution

$$x(t) \longrightarrow \boxed{y(t) = \int_{-3}^{3} x(\tau)\mathrm{d}\tau} \longrightarrow y(t)$$

(b) [10 pts.] The system above is (circle all correct choices, and explain):

(b-1) Memoryless *or* with memory. Kind of obvious that there is memory since output depends on a range of values $x(t) : -3 \le t \le 3$ of input.

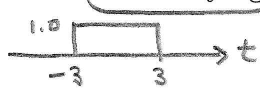
(b-2) Invertible *or* non-invertible. All I have to do is demonstrate a nonzero input signal which results in the identically zero output signal

Consider $x_1(t)$  $\Longrightarrow y_1(t) = 0 \ \forall t$ $\therefore$ system cannot be invertible.

(b-3) BIBO stable *or* unstable. For any particular input the output is a constant (dc) signal. Say $|x(t)| \le B \ \forall t$. Then

$$|y(t)| = \left| \int_{-3}^{3} x(\tau)\,d\tau \right| \le \int_{-3}^{3} |x(\tau)|\,d\tau \le 6B$$

(b-4) Time invariant *or* time varying.

Let $x_1(t) =$  $\Longrightarrow y_1(t) = 6 \ \forall t$. Define $x_2(t) = x_1(t-6)$

ie $x_2(t) = 0$ for $-3 < t < 3 \Longrightarrow y_2(t) = 0 \ \forall t$.

Note: $y_2(t) \ne y_1(t-6)$

(b-5) Linear *or* non-linear.

Sums or integrals of signals are always linear.

Problem 2. (cont'd.) Name: Solution

$$x(t) \longrightarrow \boxed{y(t) = \int_{t-3}^{t+3} x(\tau)\mathrm{d}\tau} \longrightarrow y(t)$$

(c) [10 pts.] The system above is (circle all correct choices, and explain):

(c-1) Memoryless *or* with memory. output at time t depends on a range $x(s)$: $t-3 < s < t+3$.

(c-2) Invertible *or* non-invertible. Any input which is periodic of period 6 with zero mean will produce the "all zero" output.

(c-3) BIBO stable *or* unstable. If carefully check the proof given in b-3 you will see that is goes through.

Suppose $y_1(t) = \int_{t-3}^{t+3} x_1(\tau)\,d\tau$. Then define

(c-4) Time invariant *or* time varying. $x_2(t) = x_1(t-T)$ and compute

$$y_2(t) = \int_{t-3}^{t+3} x_2(\tau)\,d\tau = \int_{t-3}^{t+3} x_1(\tau - T) \underset{\substack{\uparrow \\ s=\tau-T}}{=} \int_{t-3-T}^{t+3-T} x_1(s)\,ds = \int_{(t-T)-3}^{(t-T)+3} x_1(s)\,ds = y_1(t-T)$$

(c-5) Linear *or* non-linear.

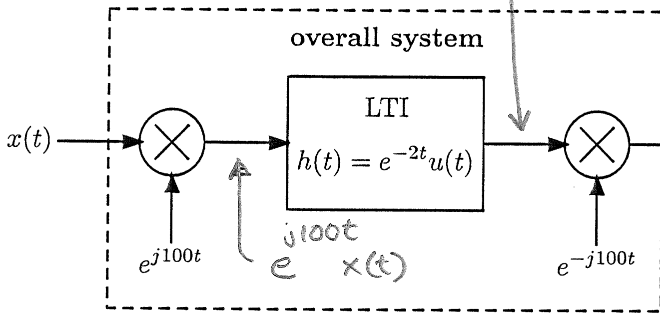
Sums or integrals of signals are always linear.

Solution

Problem 2. (cont'd.)

Name: ____________

$$= \int_0^\infty e^{-2\tau} x(t-\tau) e^{j100(t-\tau)} d\tau = e^{j100t} \int_0^\infty e^{-(2+j100)\tau} x(t-\tau) d\tau$$



$$y(t) = e^{-j100t}\{ \; \} = \int_0^\infty e^{-(2+j100)\tau} x(t-\tau)\, d\tau$$

$\therefore$ From $x \mapsto y$ it is LTI with impulse response $h_{overall}(t) = e^{-(2+j100)t} u(t)$

(d) [10 pts.] The system above is (circle all correct choices, and explain):

(d-1) Invertible or non-invertible. To show I need to demonstrate an inverse syst. ie find $h_{inv}(t)$ st. $h_{overall} * h_{inv}(t) = \delta(t)$

(d-2) Causal or non-causal. Impulse resp. is causal.

(d-3) BIBO stable or unstable. Easy to show: $\int_0^\infty \left| e^{-(2+j100)t} \right| = \int_0^\infty e^{-2t} dt < \infty$

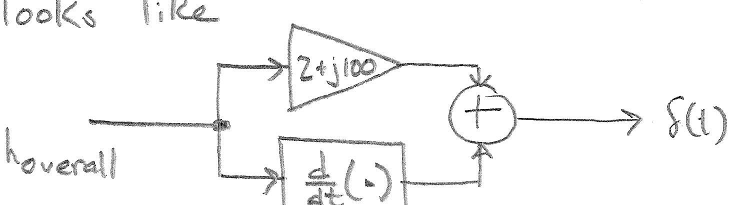
(d-4) Time invariant or time varying. Thats the "TI" in LTI.

(d-5) Linear or non-linear. The "L" in LTI.

A strategy: Try to find some linear combination of derivatives, etc. which takes $h_{overall}$ to δ.

$$\frac{d}{dt} h_{overall}(t) = -(2+j100) e^{-(2+j100)t} u(t) + e^{-(2+j100)t}\delta(t)$$
$$= -(2+j100) e^{-(2+j100)t} u(t) + \delta(t)$$

$\Rightarrow$ $(2+j100) h_{overall}(t) + \frac{d}{dt} h_{overall}(t) = \delta(t)$ $\therefore$ inverse system looks like



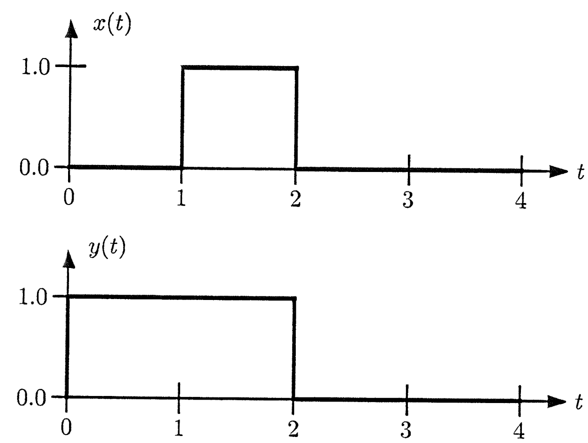
$\therefore$ $h_{inv}(t) = (2+j100)\delta(t) + \delta'(t)$

Name: Solution

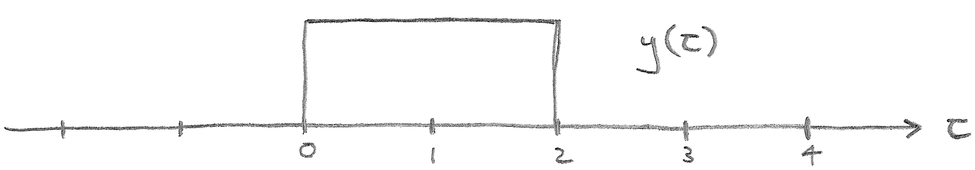
Problem 3. *Linear and Time-Invariant Systems.* [30 pts. total, **LO-ii, LO-iii**]
The problem has two parts.

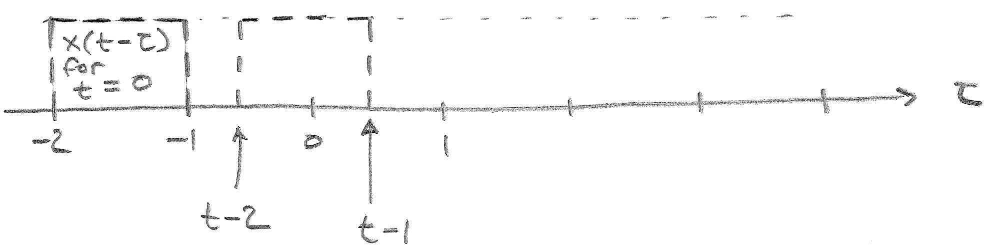
(a) [15 pts.] Compute and plot the convolution of the two rectangular pulses shown below.

Its very helpful to Know that conv. rectangular signals yields outputs with ramps in them



$$Z(t) = x * y\,(t)$$
$$= \int y(\tau)\,x(t-\tau)\,d\tau$$





$x(t-\tau)$ for

$0 < t-1 < 1 \iff 1 < t < 2$

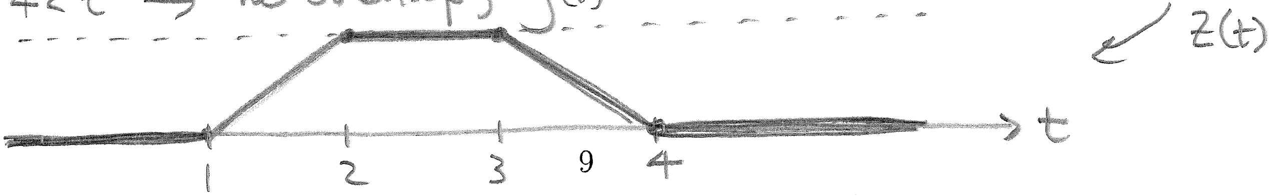
Cases: $t-1 < 0 \iff t < 1 \implies$ no overlap; $y(t) = 0$

$1 < t < 2 \implies$ partial overlap; $y(t)$ is linear with pos. slope

$2 < t < 3 \implies$ full overlap; $y(t) =$ const. $=$ area under $x(t) = 1$

$3 < t < 4 \implies$ partial overlap; $y(t) =$ linear with neg. slope

$4 < t \implies$ no overlap; $y(t) = 0$



Problem 3. (cont'd.) Name: Solution

(b) [15 pts.] Find the impulse response of the discrete-time system described by the difference equation:

$$y[n] + 2y[n-1] = x[n-1].$$

Let $x[n] = \delta[n]$

$\Longrightarrow y[n] = h[n]$ with $h[-1] = 0$ at rest before application of input.

$$h[n] + 2h[n-1] = \delta[n-1]$$

$n=0$ $\quad h[0] + 2\cancel{h[-1]}_{0} = \cancel{\delta[-1]}_{0} \Longrightarrow h[0] = 0$

$n=1$ $\quad h[1] + 2\cancel{h[0]}_{0} = \delta[0] \Longrightarrow h[1] = 1$

$n=2$ $\quad h[2] + 2h[1] = 0 \Longrightarrow h[2] = -2$

$n=3$ $\quad h[3] + 2h[2] = 0 \Longrightarrow h[3] = 4$

$n=4$ $\quad h[4] + 2h[3] = 0 \Longrightarrow h[4] = -8$

$$\therefore \quad h[n] = \begin{cases} 0 & n = 0 \\ (-2)^{n-1} & n \geq 1 \end{cases}$$

or $h[n] = (-2)^{n-1} u[n-1]$