
The following 6 problems are Dr. Davis' Aug. '03 Qualifying Exam

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1. Let f be an integer-valued function on $\mathbb{R}$. show that $\{x : f \text{ is not continuous at } x\}$ is a Borel set.

2. Let A and B be (not necessarily Lebesgue measurable) subsets of $\mathbb{R}$ and let $|\ |_e$ stand for Lebesgue outer measure. Prove that if $|A|_e = 1$ and $|B|_e = 1$ and $|A \cup B|_e = 2$ the $|A \cap B|_e = 0$.

3. Show, with proof, $F'(1) < 0$, where
$$F(x) = \int_0^\infty \frac{e^{-xy}}{y^3+1} dy.$$

4. Prove that if f is a uniformly continuous function on $\mathbb{R}$ and if $h(x) = \int_{x-1}^{x+1} f(s)ds$ then h is uniformly continuous on $\mathbb{R}$.

5. Let $g(x)$ be a continuous function on $[0, 1]$ satisfying $g(0) = 0, g(1) \leq 1$, and $g(s) \leq g(t)$ if $0 \leq s < t \leq 1$. Put $\phi_n(x) = g(x)^n$. Prove that it f is a continuous function on $[0, 1]$, $\lim_{n\to\infty} \int_0^1 f(x)d\phi_n(x)$ exists.

6. Let f be a bounded Lebegue measurable function on $\mathbb{R}$. Put $g(x) = \sup\{a \in \mathbb{R} : |\{y : y \in (x, x+1) \text{ and } f(y) > a\}| > 0\}$, where $|\ |$ is Lebesgue measure (i.e. $g(x)$ equals the essential supremum of f over $(x, x+1)$). Prove $\liminf_{x\to 0} g(x) \geq g(0)$.

The following problems were written by Dr. Davis for an exam

1. Let f_n be a sequence of measurable functions with $f_n \to 0$ a.e. on $[a, b]$. Show there exists a subsequence f_{n_j} such that $\sum_j f_{n_j}(x)$ converges a.e. in $[a, b]$.

2. Fix $x_0 \in \mathbb{R}$. Show $\{(x_0, y)|y \in \mathbb{Q}\} \in \mathcal{B}$, the Borel sets on $\mathbb{R}^2$.

3. Let K be a compact set in $\mathbb{R}^n$ and $E \subset \mathbb{R}^n$ that is disjoint from K, (E need not be measurable). Let $|\_|_e$ denote Lebesgue outer measure, and show
$$|K \cup E|_e = |K| + |E|_e$$
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