Convergence of the CTFT

$x(t) \overset{CTFT}{\longleftrightarrow} H(\omega)$ $\quad \mathcal{F}\{x(t)\} = H(\omega)$

- Dirichlet conditions
  1. $x(t)$ is absolutely integrable

     $$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

  2. $x(t)$ must have a finite number of maxima and minima in any finite period.
  3. $x(t)$ must have a finite number of discontinuities in a finite. These discontinuities must be finite.

- Periodic signals (cos, sine) are not absolutely integrable. However they have FS representations. Combine the two in one framework using impulse functions.

Ex $\quad x(t) = \cos\omega_0 t = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty}\left(\frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}\right)e^{-j\omega t}dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t}dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-j(\omega+\omega_0)t}dt$$

$$= \frac{1}{2}\cdot\frac{-1}{j(\omega-\omega_0)}\left[e^{-j(\omega-\omega_0)t}\right]_{-\infty}^{\infty} + \frac{1}{2}\cdot\frac{-1}{j(\omega+\omega_0)}\left[e^{-j(\omega+\omega_0)t}\right]_{-\infty}^{\infty}$$

Does not obviously converge.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

$$X(\omega) = \pi(\delta(\omega-\omega_0) + \delta(\omega+\omega_0))$$

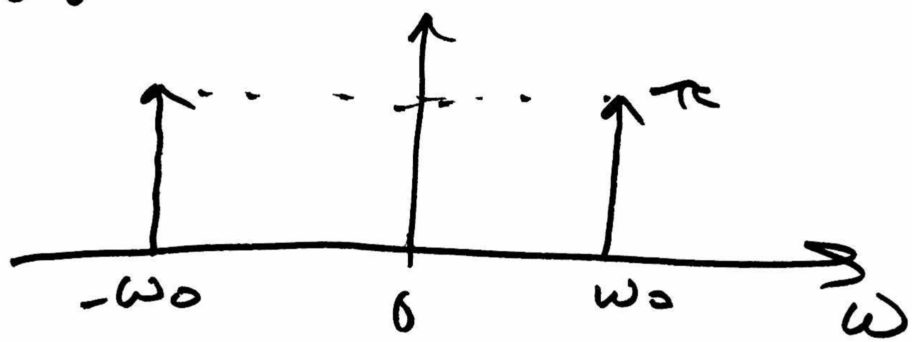
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\pi\,\delta(\omega-\omega_0)e^{j\omega t}d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty}\pi\,\delta(\omega+\omega_0)e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\cdot\pi e^{j\omega_0 t}\int_{-\infty}^{\infty}\delta(\omega-\omega_0)d\omega + \frac{1}{2\pi}\cdot\pi e^{-j\omega_0 t}\int_{-\infty}^{\infty}\delta(\omega+\omega_0)d\omega$$

$$= \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t} = \cos\omega_0 t$$

Review: Analysis (Fourier integral) wouldn't converge directly, so we looked synthesis, guessed the $X(\omega)$, validated it.

$$\cos\omega_0 t \xleftrightarrow{CTFT} \pi\left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right)$$



<u>Ex</u> Connecting FS with CTFT

$$\mathcal{F}^{-1}\{\delta(\omega-\omega_0)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega-\omega_0)e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}e^{j\omega_0 t}$$

$$X(\omega) = \sum_{k=-\infty}^{\infty}(a_k \cdot 2\pi)\delta(\omega-\omega_0 k)$$

$$x(t) = \mathcal{F}^{-1}\Big\{\sum_{k=-\infty}^{\infty}(a_k)\big(2\pi\,\delta(\omega-\omega_0 k)\big)\Big\}$$

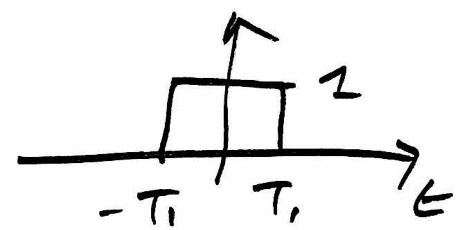
$$= \sum_{k=-\infty}^{\infty} a_k\,\mathcal{F}^{-1}\{2\pi\,\delta(\omega-\omega_0 k)\}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t}$$ ← synthesis equation for the FS

If a signal has FS representation ($x(t) \overset{FS}{\longleftrightarrow} a_k$), then the CTFT is an impulse train weighted by $2 \cdot \pi \cdot a_k$.

Periodicity in one domain implies discreteness in the other domain.

<u>Ex</u> Periodic square wave



period T
$\omega_0 = \frac{2\pi}{T}$

$$x(t) \longleftrightarrow a_k = \frac{\sin k\omega_0 T_1}{\pi k}$$

$$\mathcal{F}\{x(t)\} = \sum_{k=-\infty}^{\infty} a_k \cdot 2\pi \, \delta(\omega - \omega_0 k)$$

$$= \sum_{k=-\infty}^{\infty} 2 \frac{\sin k\omega_0 T_1}{\pi k} \delta(\omega - \omega_0 k)$$

Properties of CTFT

- Linearity

$$\mathcal{F}\{a x(t) + b y(t)\} = a\,\mathcal{F}\{x(t)\} + b\,\mathcal{F}\{y(t)\}$$
$$= a X(\omega) + b Y(\omega)$$

- Time shifting

$$\mathcal{F}\{x(t - t_0)\} = e^{-j\omega t_0}\,\mathcal{F}\{x(t)\}$$
$$= e^{-j\omega t_0} X(\omega)$$

* Time shifts don't affect the magnitude of the CTFT, just add a phase term.

$$\cancel{|e^{-j\omega t_0}|} \quad |e^{-j\omega t_0} X(\omega)| = |X(\omega)|$$

- Time scaling

$$\mathcal{F}\{x(at)\} = \frac{1}{|a|} X\left(\frac{\overset{\omega}{f}}{a}\right) \qquad \text{real } a \neq 0$$

  - compression in one domains stretches the other domain