7 FEBRUARY

Art of Counting

Intro: The Pigeonhole Principle

Theorem: If you have N objects ($N \geq 0$) and you sort them into K categories, then at least one category will have at least $\lceil \frac{N}{K} \rceil$* objects.

* $\lceil \lambda \rceil$ = smallest integer ~~for~~ t for which $\lambda \leq t$ ("ceiling")

$\Rightarrow$ Intuitive and may seem trivial. but one w/ useful applications

Eg.

From list $\{1, \ldots, 2N\}$ pick $N+1$ numbers $\{a_1, \ldots, a_{n+1}\}$. Show that for some i, j a_i divides a_j.

Idea

$\rightarrow$ Factor $a_i = 2^{n_i} \cdot q_i$ where q_i is odd.

$\rightarrow$ The $N+1$ quotients, $q_1, \ldots, q_{N+1}$ are odd and none exceeds $2N$.

$\rightarrow$ There are N odd integers in $2N$: $\{1, 3, 5, \ldots, 2N-1\}$

$\rightarrow$ Pigeonhole says: $N+1$ objects $q_1, \ldots, q_{N+1}$, when placed in bucket of N odd numbers $1, 3, \ldots, 2N-1$ gives at least one bucket w/ at least $\lceil \frac{N+1}{N} \rceil = 2$ objects.

$\Rightarrow$ at least two qs are the same.

$q_i = q_j$.

Then, $a_i = 2^{n_i} q_i$ and $a_j = 2^{n_j} q_j$ becomes $a_i = 2^{n_i} q$ $a_j = 2^{n_j} \cdot q$.

If $n_i > n_j$, 2^{n_i} is a multiple of 2^{n_j} so $2^{n_i} q_i$ is mult of $2^{n_j} q_j$

If $n_i < n_j$, 2^{n_i} is a divisor of 2^{n_j} so $2^{n_i} q_i$ is divisor of $2^{n_j} q_j$.

ADVANCED COUNTING

Recall: "Permutation" of n objects is one way ~~to~~ of arragment in order

There are $n!$ such permutations

Def An r-permutation of n objects is an ordered subcollection of n-elements.

Special case: an n-permutation of n object is simply permutation of n

The collection of all r-permutation of n object has

$$n \cdot (n-1) \cdot \ldots \cdot (n-r+1) = P(n, r)$$

Def An r-combination on n objects is a subcollection of size r (NOT ordered)

The number of r-combinations on n object is given by

$$\frac{P(n,r)}{r!} \begin{array}{l} \leftarrow \text{all possible to pick } r \text{ things from } n \\ \leftarrow \text{possible ways to arrange } r \text{ things} \end{array} \Big\} \text{ gives number of distinct clusters.}$$

$$= C(n,r)$$

$$= \binom{n}{r}$$

Lemma (mini-theory)

$$\binom{n}{r} = \binom{n}{n-r}$$

→ Two Proofs

(a) Algebraic ("stupid") proof

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{1\cdot 2\cdot \cdots \cdot r} \underset{=}{\overset{\text{if } r<(n-r)}{=}} \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)\cdots(r+1)}{1\cdot 2\cdots \cdot r\cdot(r+1)\cdots(n-r)} = \binom{n}{n-r}$$

$$\binom{n}{n-r} = \frac{n(n-1)(n-2)\cdots(n-(n-r)+1)}{1\cdot 2\cdot \cdots \cdot (n-r)} \overset{\text{if } r>(n-r)}{=} \frac{n(n-1)\cdots(r+1)(r-1)\cdots(n-r+1)}{1\cdot 2\cdots(n-r)\cdot(n-r+1)\cdots\cdot r} = \binom{n}{r}$$

So $\binom{n}{r} = \binom{n}{n-r}$ by explicit computation

(b) "Clever" proof

$\binom{n}{r}$ = number of ways of choosing r things from n

= number of ways of leaving n−r things on the table

= (changing perspective) number of ways of taking n−r from n.

e.g.

Divide n candies between two brothers. Older taking r candies is equivalent to the younger taking n−r candies.

Lemma 2

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

(1) algebraic proof

need to show: $\dfrac{n!}{r!(n-r)!} + \dfrac{n!}{(r+1)!(n-r-1)!} = \dfrac{(n+1)!}{(r+1)!(n-r)!}$

LHS: → we mult each ↗ ↗ by particular mult so the each denominator beomes $(r+1)!\,(n-r)!$

$$= \frac{n!(r+1) + n!(n-r)}{r!(n-r)!} = n!(r+1+n-r) = n!(n+1) = \frac{(n+1)!}{r!\,(n-r)!} = \text{RHS} \checkmark$$

(2) Combinatorial Proof

task: pick $r+1$ things from $n+1$ objects

Call $n+1$ things $x_1, x_2, \ldots, x_n, y$ - $n+1$th obj.

two option for choosing:

with y: $\binom{n}{r}$ options of x_s

without y: $\binom{n}{r+1}$ options of x_s

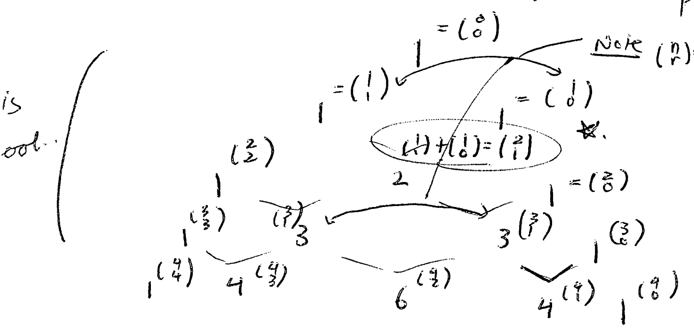
So ~~choosing~~ total number of ways:

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

Connecting to Pascals Triangle

i) Intial condition: Edges of pascal's triangle will have valus $1 = \binom{n}{0}$

Note $\binom{n}{r} = \binom{n}{n-r}$: symmetry.



Triangle Numbers obey the same rule ...

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

Morally, this is equivalent of saying Pascal's Triangle #s and "choose numbers" are ~~from~~ the same diff. eq.

The fact that the boundary of the triangle of both sets of ~~agree~~ numbers agree forces the two sets to agree everywhere

(same init cond + same mechnism = sam everywhere)

EXPERIMEN

raise $(x+y)$ to various powers

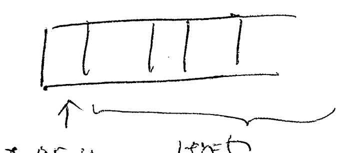
$$(x+y)^0 = \cancel{0}\ 1$$
$$(x+y)^1 = x+y$$
$$(x+y)^2 = x^2 + \boxed{2xy} + y^2$$
$$(x+y)^3 = x^3 + \boxed{3x^2y + 3xy^2} + y^3$$
] prof disagrees...?

However,

$$(x+y)^2 = (x+y)(x+y) = xx + \boxed{xy + yx} + yy$$
$$(x+y)^3 = (x+y)(x+y)(x+y) = xxx + \boxed{xxy + xyx + yxx} + \boxed{xyy + yxy + yyx} + yyy$$

grouping by equal #s of x and y factors produces the coefficients...

What is the coefficient that counts the monomial of n degree that uses "x" r times and "y" $n-r$ times?



fill in r xs...

fill rest with ys

$\Longrightarrow$ equivalent of # of ways of filling in xs since ys spots are determined upon selections of xs...

$\Rightarrow$ coefficient counts the number of ways to pick r locations that would house the "x_s"

$= \binom{n}{r}$

Thm Binomial Thm

$$(x+y)^n = \binom{n}{n}x^n + \binom{n}{n-1}x^{n-1}y + \dots + \binom{n}{n-i}x^{n-i}y^i + \dots \binom{n}{0}y^n$$

Special Instances

1. $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n$

Proof: if $x = y = 1$, by Binomial Thm

$$(1+1)^n = 2^n = \binom{n}{n} + \binom{n}{n-1} + \dots \binom{n}{0} \quad ✓$$