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ECE 301 Signals and Systems
Homework # 5 Solution

4.34 a) $H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega + 4}{6 - \omega^2 + 5j\omega}$.

$$\therefore \left[6 + (j\omega)^2 + 5j\omega\right] Y(j\omega) = (j\omega + 4) X(j\omega)$$

$$6Y(j\omega) + (j\omega)^2 Y(j\omega) + 5j\omega Y(j\omega) = j\omega X(j\omega) + 4X(j\omega).$$

$$\Updownarrow \text{ F.T.}$$

$$\boxed{\frac{d^2}{dt^2} y(t) + 5\frac{d}{dt} y(t) + 6y(t) = \frac{d}{dt} x(t) + 4x(t)}$$

b) The impulse response of the system is simply the inverse transform of $H(j\omega)$.

$$H(j\omega) = \frac{j\omega + 4}{(2+j\omega)(3+j\omega)} = \frac{2}{2+j\omega} - \frac{1}{3+j\omega}.$$

$$\Updownarrow \text{ F.T.}$$

$$\boxed{h(t) = 2e^{-2t}u(t) - e^{-3t}u(t)}$$

c)

$$x(t) = e^{-4t}u(t) - te^{-4t}u(t) \xleftrightarrow{\text{F.T}} X(j\omega) = \frac{1}{4+j\omega} - \frac{1}{(4+j\omega)^2}$$

$$Y(j\omega) = X(j\omega)H(j\omega)$$

$$= \left[\frac{1}{4+j\omega} - \frac{1}{(4+j\omega)^2}\right]\left[\frac{2}{2+j\omega} - \frac{1}{3+j\omega}\right]$$

$$= \frac{1}{(4+j\omega)(2+j\omega)} = \frac{1}{2(2+j\omega)} - \frac{1}{2(4+j\omega)}$$

$$y(t) = \frac{1}{2}e^{-2t}u(t) - \frac{1}{2}e^{-4t}u(t).$$

O+W 4.42

$$g_1(t) = \left[x(t)\cos\omega_0 t\right] * h(t)$$
$$g_2(t) = \left[x(t)\sin\omega_0 t\right] * h(t)$$

where

$$x(t) = \sum_k a_k e^{j100kt}$$ is real-valued, periodic

and $h(t)$ is impulse resp. of stable LTI system.

(a) Find value for ω_0 and necc. constraints on $H(j\omega)$ to ensure that

$$g_1(t) = \mathrm{Re}(a_5) \qquad g_2(t) = \mathrm{Im}(a_5).$$

$$\cos\omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$\Rightarrow x(t)\cos\omega_0 t = \frac{1}{2}\sum_k a_k e^{j(\omega_0 + 100k)t} + \frac{1}{2}\sum_k a_k e^{j(-\omega_0 + 100k)t}$$

Notice that $\mathrm{Re}(a_k) = \frac{a_k + a_k^*}{2} = \frac{a_k + a_{-k}}{2}$

(since $x(t)$ is real-valued implying that $a_k^* = a_{-k}$).

Now reorder the sum above s.t.

$x(t)\cos\omega_0 t$

$$= \frac{1}{2}\sum_k a_{-k} e^{j(\omega_0 - 100k)t} + \frac{1}{2}\sum_k a_k e^{j(-\omega_0 + 100k)t}$$

$$x(t)\cos\omega_0 t = \frac{1}{2}\sum_k \left\{ \left(a_k e^{-j(\omega_0-100k)t}\right)^* + a_k e^{-j(\omega_0-100k)t} \right\}$$

$$= \sum_k \mathrm{Re}\left(a_k e^{-j(\omega_0-100k)t}\right)$$

$$= \sum_k |a_k| \cos\left[(\omega_0-100k)t + \angle a_k\right]$$

This is a sum of cosinusoids of frequencies $\omega_0 - 100k$. If we pick

$$\omega_0 = 500$$

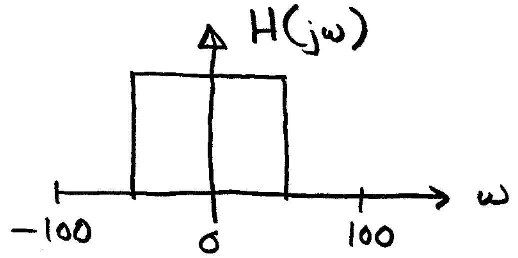
then the DC term in the sum will be $k=5$ and is

$$\underset{\text{term}}{\text{DC}}\left\{x(t)\cos(500t)\right\} = |a_5|\cos\angle a_k$$
$$= \mathrm{Re}(a_5).$$

Can use any LTI filter $H(j\omega)$ which has the property

$$H(j\omega) = \begin{cases} 1 & \omega = 0 \\ 0 & \omega = 100m \quad m = \pm 1, \pm 2, \dots \end{cases}$$

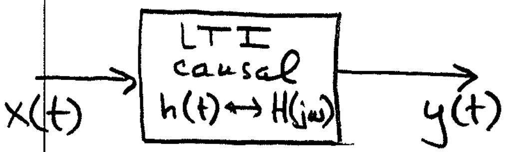
for example



$\longleftarrow$ (b).

An analogous argument works for $g_2(t)$.

O+W 4.44



$$\frac{dy}{dt} + 10y(t) = \int_{-\infty}^{\infty} x(\tau) z(t-\tau)\, d\tau - x(t).$$

where $z(t) = e^{-t}u(t) + 3\delta(t)$.

(a) Find $H(j\omega)$.

Take F.T. of equation above and use fact that the integral is a convolution

$$j\omega Y(j\omega) + 10\,Y(j\omega) = X(j\omega)Z(j\omega) - X(j\omega)$$

$$Y(j\omega)\left[j\omega + 10\right] = X(j\omega)\left[Z(j\omega) - 1\right]$$

$$\Longrightarrow \frac{Y(j\omega)}{X(j\omega)} = H(j\omega) = \frac{Z(j\omega) - 1}{j\omega + 10}$$

To finish need transform of $Z(j\omega)$, which can get from Tables

$$\delta(t) \longleftrightarrow 1$$

$$e^{-t}u(t) \longleftrightarrow \frac{1}{1+j\omega}$$

$$\therefore\ Z(j\omega) = \frac{1}{1+j\omega} + 3 \Longrightarrow Z(j\omega) - 1$$

$$= \frac{1}{1+j\omega} + 2$$

$$= \frac{1 + 2 + 2j\omega}{1+j\omega}$$

$$= \frac{3 + 2j\omega}{1+j\omega}$$

$$\therefore \quad \frac{Y(j\omega)}{X(j\omega)} = \frac{3 + 2j\omega}{(10+j\omega)(j\omega+1)}$$

(b) To find causal $h(t)$ with this Fourier transform, do a partial fraction expansion and use transform tables

$$H(v) = \frac{2v+3}{(v+10)(v+1)} = \frac{A}{v+10} + \frac{B}{v+1}$$

$v = j\omega$

$$A = \left.\frac{2v+3}{v+1}\right|_{v=-10} = \frac{-17}{-9} = \frac{17}{9}$$

$$B = \left.\frac{2v+3}{v+10}\right|_{v=-1} = \frac{1}{9}$$

$$\therefore \quad H(j\omega) = \frac{17/19}{j\omega+10} + \frac{1/9}{j\omega+1}$$

$$h(t) = \frac{17}{19} e^{-10t} u(t) + \frac{1}{9} e^{-t} u(t)$$

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O+W 4.51

(a) LTI systems with $h(t)$ and $g(t)$ which are inverses of one another ie

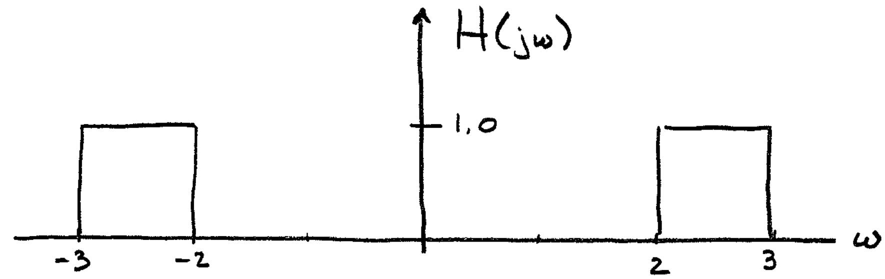
$$h * g(t) = \delta(t)$$

Taking Fourier transform of equation above

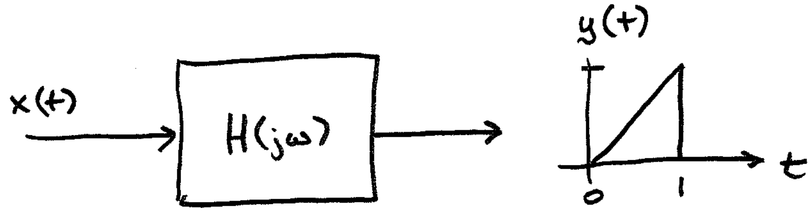
$$H(j\omega)\,G(j\omega) = 1$$

$$\Rightarrow G(j\omega) = 1/H(j\omega)$$

(b)



(b-i) Is there $x(t)$ st.



This is impossible. The reason is that any output of system $h(t) \leftrightarrow H(j\omega)$ can only have nonzero spectrum over the range $2 < |\omega| < 3$, ie must be bandlimited as $H(j\omega)$ is. But $y(t) \leftrightarrow Y(j\omega)$ is not bandlimited like this (check this!)

(b-ii) System is not invertible. An invertible system must be able to generate any signal at its output.

(c)

$$h(t) = \sum_{k=0}^{\infty} e^{-kT}\,\delta(t-kT) \quad \text{is impulse resp. of echo.}$$

To remove effect of echo must find an inverse system ie $g(t)$ st.

$$h * g(t) = \delta(t) \longleftrightarrow 1 = H(j\omega)\,G(j\omega)$$

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$$H(j\omega) = \int h(t) e^{-j\omega t}\, dt = \int \sum_{k=0}^{\infty} e^{-kT} \delta(t-kT)\, e^{-j\omega t}\, dt$$

$$= \sum_{k=0}^{\infty} e^{-kT} \int \delta(t-kT)\, e^{-j\omega t}\, dt = \sum_{k=0}^{\infty} e^{-kT} e^{-j\omega kT}$$

$$= \frac{1}{1 - e^{-T} e^{-j\omega T}} \qquad \left(\text{note } \left| e^{-T} e^{-j\omega T} \right| < 1\right)$$

$$\therefore\ G(j\omega) = \frac{1}{H(j\omega)} = 1 - e^{-T} e^{-j\omega T}$$

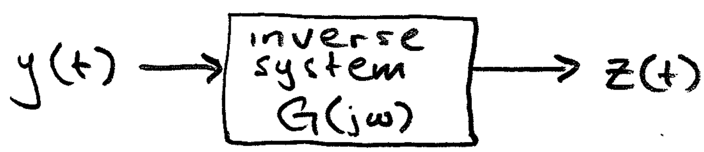
(d) Find diff. eqn. for inverse of system with impulse response $h(t) = 2\delta(t) + u_1(t)$.

Recall $u_1(t) = \delta'(t)$ (e.g., the unit doublet defined as on page 132 of O+W). Using FT properties

$$H(j\omega) = 2 + j\omega \implies \text{inverse system}$$

$$G(j\omega) = \frac{1}{2 + j\omega}$$

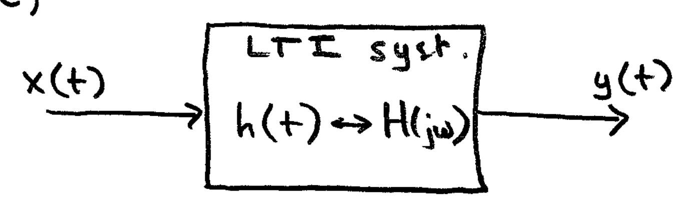
In the usual way we find that if



then

$$\frac{dz}{dt} + 2z(t) = y(t).$$

(e)



$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y(t) = \frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x(t)$$

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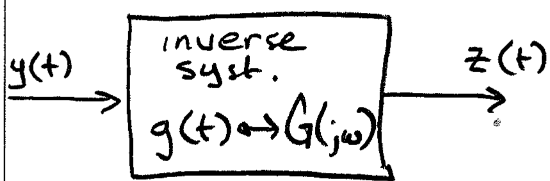
Clearly

$$H(j\omega) = \frac{(j\omega)^2 + 3(j\omega) + 2}{(j\omega)^2 + 6(j\omega) + 9}$$

The inverse system must have transfer function

$$G(j\omega) = \frac{(j\omega)^2 + 6(j\omega) + 9}{(j\omega)^2 + 3(j\omega) + 2}$$

Therefore the differential equation satisfied by the inverse system is



$$\frac{d^2z}{dt^2} + 3\frac{dz}{dt} + 2z(t) = \frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y(t)$$

(it is of exactly same form as the original differential equation — the only difference between them is in the interpretation of which time function is input and which is output).

To find corresponding impulse responses we will need to do a partial fraction expansion and use the transform tables. Let $v = j\omega$.

$$H(v) = \frac{v^2 + 3v + 2}{v^2 + 6v + 9} \qquad G(v) = \frac{v^2 + 6v + 9}{v^2 + 3v + 2}$$

$$v^2 + 6v + 9 = (v+3)^2 \qquad v^2 + 3v + 2 = (v+1)(v+2)$$

But before we do the partial fraction expansion we need to do long divisions to make numerator degrees smaller than denominator degrees.

$$H(v) = 1 - \frac{3v+7}{(v+3)^2}$$

do PFE

$$= \frac{A}{v+3} + \frac{B}{(v+3)^2} \qquad B = 3v+7\Big|_{v=-3} = -2$$

$$\frac{3v+7}{(v+3)^2} = \frac{A}{v+3} - \frac{2}{(v+3)^2} \Rightarrow 3v+7 = A(v+3) - 2$$

$$3v+7 = Av + (3A-2)$$

$$\therefore A = 3$$

$$\therefore H(v) = 1 - \frac{3}{v+3} + \frac{2}{(v+3)^2}$$

$$H(j\omega) = 1 - \frac{3}{3+j\omega} + \frac{2}{(3+j\omega)^2}$$

$$\updownarrow$$

$$h(t) = \delta(t) - 3e^{-3t}u(t) + 2te^{-3t}u(t)$$

$$G(v) = 1 + \frac{3v+7}{(v+1)(v+2)}$$

do PFE

$$= \frac{A}{v+1} + \frac{B}{v+2} \qquad A = \frac{3v+7}{v+2}\Big|_{v=-1} = \frac{4}{1} = 4$$

$$B = \frac{3v+7}{v+1}\Big|_{v=-2} = \frac{1}{-1} = -1$$

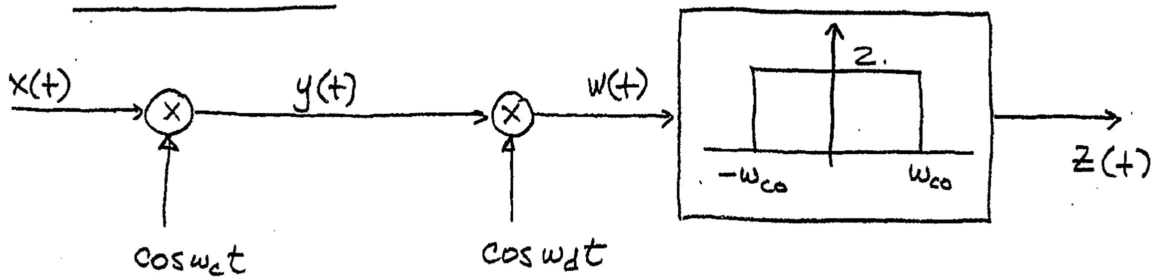
$$\therefore G(v) = 1 + \frac{4}{v+1} - \frac{1}{v+2}$$

$$G(j\omega) = 1 + \frac{4}{1+j\omega} - \frac{1}{2+j\omega}$$

$$\updownarrow$$

$$g(t) = \delta(t) + 4e^{-t}u(t) - e^{-2t}u(t).$$

O+W 8.23



(a) $w(t) = x(t)\cos\omega_c t\,\cos\omega_d t$

$$= \tfrac{1}{2}x(t)\left[\cos(\omega_d-\omega_c)t + \cos(\omega_d+\omega_c)t\right]$$

$$= \tfrac{1}{2}x(t)\cos\Delta\omega t + \tfrac{1}{2}x(t)\cos\left(2\omega_c+\Delta\omega\right)t$$

Assume that $x(t)$ is bandlimited to ω_M and ω_{co} satisfies

$$\omega_M + \Delta\omega < \omega_{co} < 2\omega_c + \Delta\omega - \omega_M$$

Then it will follow that only the low freq. term above makes it through the LPF; hence

$$z(t) = x(t)\cos\Delta\omega t$$

(picture of part (b) makes this clear).

(b) From the modulation property and the transform of cosine;

$$W(j\omega) = \tfrac{1}{4}X(j(\omega-\Delta\omega)) + \tfrac{1}{4}X(j(\omega+\Delta\omega))$$
$$+ \tfrac{1}{4}X(j(\omega-2\omega_c-\Delta\omega)) + \tfrac{1}{4}X(j(\omega+2\omega_c+\Delta\omega))$$