Homework 5

4.23) $$x_0(t) = \begin{cases} e^{-t} & 0 \le t \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$X_0(j\omega) = \int_{-\infty}^{\infty} x_0(t)\, e^{-j\omega t}\, dt$$

$$= \int_0^1 e^{-t} e^{-j\omega t}\, dt$$

$$= \int_0^1 e^{-t(1+j\omega)}\, dt$$

$$= \left. \frac{e^{-t(1+j\omega)}}{-(1+j\omega)} \right]_0^1$$

$$= \frac{1 - e^{-(1+j\omega)}}{1+j\omega}$$

(a) $$x_1(t) = x_0(-t) + x_0(t)$$

$$X_1(j\omega) = x_0(-j\omega) + x_0(j\omega)$$

$$= \frac{1 - e^{-(1-j\omega)}}{1-j\omega} + \frac{1 - e^{-(1+j\omega)}}{1+j\omega}$$

$$= \frac{2 - e^{-(1-j\omega)} - j\omega e^{-(1-j\omega)} - e^{-(1+j\omega)} + j\omega e^{-(1+j\omega)}}{1+\omega^2}$$

$$= \frac{2 - 2e^{-1}\cos(\omega) - 2\omega e^{-1}\sin(\omega)}{1+\omega^2}$$

(b) $x_2(t) = -x_0(-t) + x_0(t)$

$$X_2(j\omega) = X_0(j\omega) - X_0(-j\omega)$$

$$= \frac{1 - e^{-(1+j\omega)}}{1+j\omega} - \left[\frac{1 - e^{-(1-j\omega)}}{1-j\omega}\right]$$

$$= \frac{\left[1 - j\omega - e^{-(1+j\omega)} + j\omega e^{-(1+j\omega)} - \left\{1 + j\omega - e^{-(1-j\omega)} - j\omega e^{-(1-j\omega)}\right\}\right]}{(1+\omega^2)}$$

$$= \frac{j\left[-2\omega + 2e^{-1}\sin\omega + 2\omega e^{-1}\cos\omega\right]}{(1+\omega^2)}$$ Ans.

(c) $x_3(t) = x_0(t+1) + x_0(t)$

$$X_3(j\omega) = X_0(j\omega)e^{+j\omega(1)} + X_0(j\omega)$$

$$= X_0(j\omega)\left[e^{+j\omega} + 1\right]$$

$$= \left[\frac{1 - e^{-(1+j\omega)}}{1+j\omega}\right]\left[e^{j\omega} + 1\right]$$ Ans.

(d) $x_4(t) = t\, x_0(t)$

$$X_4(j\omega) = j \frac{d}{d\omega} X_0(j\omega)$$

$$= j \frac{d}{d\omega}\left[\frac{1 - e^{-(1+j\omega)}}{1+j\omega} \right]$$

$$= j\left[\frac{(1+j\omega)(-j)e^{-(1+j\omega)} + j\left[1 - e^{-(1+j\omega)}\right]}{(1+j\omega)^2} \right]$$

$$= \frac{1 - 2e^{-1}e^{-j\omega} - j\omega e^{-1} e^{-j\omega}}{(1+j\omega)^2}$$ Ans

4.24 a) 1.) $Re\{X(j\omega)\} = 0$

$X(j\omega)$ is purely imaginary

i.e. $x(t)$ is odd

$\therefore$ (a) & (d). Ans

2.) $Im\{X(j\omega)\} = 0$

$X(j\omega)$ is purely real

i.e. $x(t)$ is even

$\therefore$ (e) & (f). Ans

3.) $e^{j\alpha\omega} X(j\omega)$ is real

$x(t-t_0) \xrightarrow{\mathcal{F}} e^{-j\omega t_0} X(j\omega)$

$x(t+\alpha) \xrightarrow{\mathcal{F}} e^{j\omega\alpha} X(j\omega)$

$\downarrow$

Even

$\therefore$ (a), (b), (e) & (f). Ans:

4.) $\int_{-\infty}^{\infty} X(j\omega)\, d\omega = 0$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, d\omega$$

$$= \frac{1}{2\pi}\left[0 \right] = 0$$

i.e $x(t)$ has to pass through the origin

$\therefore$ (a), (b), (c), (d) & (f). Ans:

5.) $\int_{-\infty}^{\infty} \omega X(j\omega)\, d\omega = 0$

Area under '$\omega X(j\omega)$' $= 0$

$$\frac{d}{dt} x(t) = j\omega X(j\omega)$$

$$= 0 \quad \text{for } t = 0$$

$\therefore$ $x(t)$ is should have a slope of '0' at time zero.

$\therefore$ (b), (c), (e) & (f). Ans.

6.) $X(j\omega)$ is periodic.

i.e. $x(t) = e^{-j\omega_0 t}$

$X(j\omega) = 2\pi\,\delta(\omega - \omega_0)$

$\therefore$ (b). Ans.

4.25) (a) $y(t) = x(t+1)$

{ $y(t)$ is real & even Hence, $Y(j\omega)$ is even & real too }.

$Y(j\omega) = e^{+j\omega(1)}\,X(j\omega)$

$\measuredangle Y(j\omega) = 0$ { Real & even }

$\measuredangle X(j\omega) = -\omega$. Ans.

(b) $X(j0)$

$X(j\omega) = \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt$

$X(j0) = \int_{-\infty}^{\infty} x(t)\,e^{-j(0)t}\,dt$

$= \int_{-\infty}^{\infty} x(t)\,dt$

$$X(j0) = \int_{-1}^{0} 2\,dt + \int_{0}^{1} (-t+2)\,dt + \int_{1}^{2} t\,dt + \int_{2}^{3} 2\,dt$$

$$= 2 + 3/2 + 3/2 + 2$$

$$= 2 + 2 + 3 = \underline{7}$$

(c) $$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega$$

$$x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, d\omega$$

$$\int_{-\infty}^{\infty} X(j\omega)\, d\omega = 2\pi\, x(0)$$

$$= 2\pi(2) = 4\pi$$

(d)

(e) $\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = 2\pi \int_{-\infty}^{\infty} |x(t)|^2 dt$

$\{$ Parseval's relation $\}$

$= 2\pi \left[(2)^2 + (2.25)^2 + (2)^2 \right]$

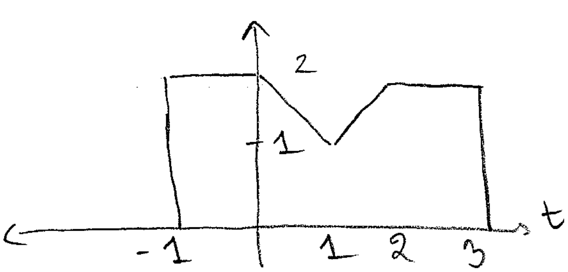
$= 2\pi (4 + 4.50 + 4)$

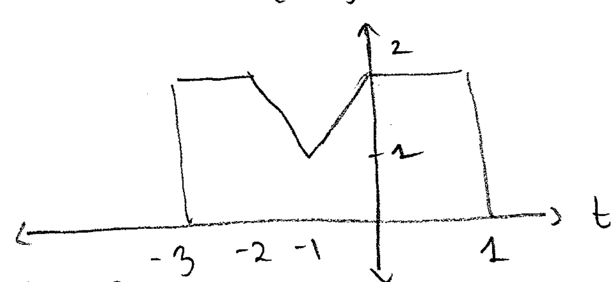
$= 2\pi (12.50)$

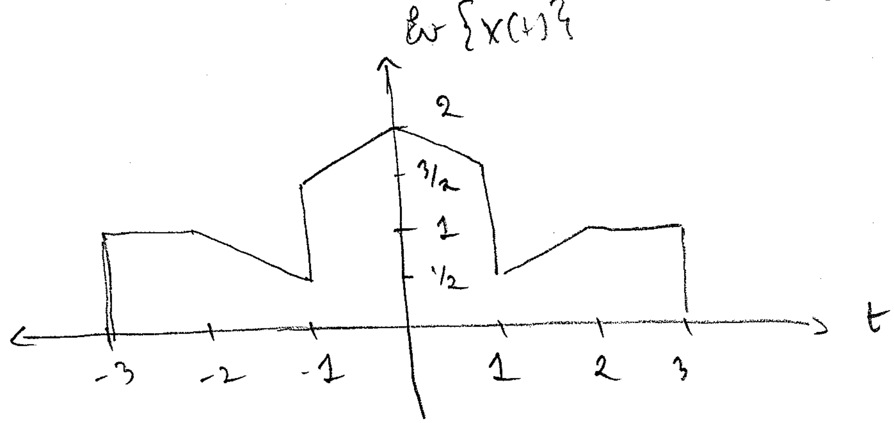
$\simeq 2\pi(13) \simeq \underline{\underline{26\pi}}$

(f) $\mathrm{Re}\{X(j\omega)\} = \mathcal{E}v\{x(t)\}$

$= \dfrac{x(t) + x(-t)}{2}$







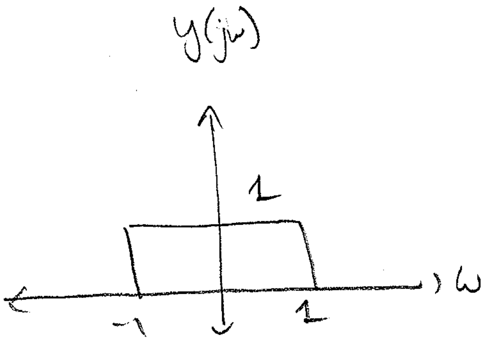
4.10) (a) $x(t) = t\left(\dfrac{\sin t}{\pi t}\right)^2$

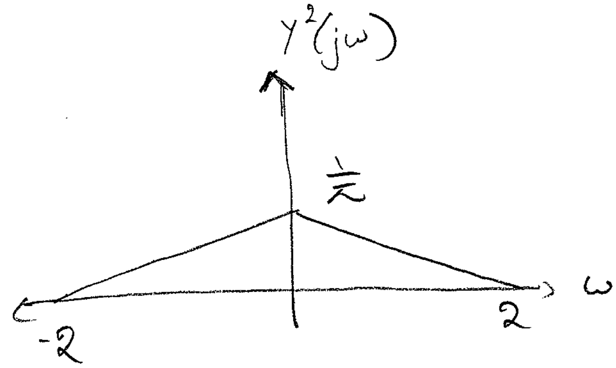
Let $y(t) = \dfrac{\sin t}{\pi t}$

$$Y(j\omega) = \begin{cases} 1 & |\omega| < 1 \\ 0 & |\omega| > 1 \end{cases} \quad \text{(Table 4.2)}$$

$$y^2(t) = \left(\frac{\sin t}{\pi t}\right)^2$$

$$\mathcal{F}[y^2(t)] = \frac{1}{2\pi}\left[Y(j\omega) * Y(j\omega)\right]$$





$\therefore\ x(t) = t\, y^2(t)$

Let $y^2(t) = w(t)$

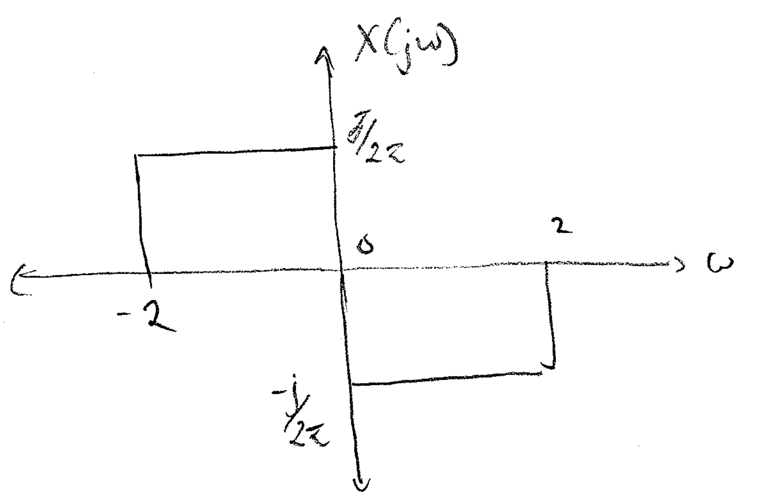
$$x(t) = t\, w(t)$$

$$X(j\omega) = j\frac{d}{d\omega}\{W(j\omega)\}$$

$$\frac{d}{d\omega}\{W(j\omega)\} = \begin{cases} \frac{1}{2\pi} & -2<\omega<0 \\ \frac{-1}{2\pi} & 0<\omega<2 \\ 0 & |\omega| > 2 \end{cases}$$

$$\therefore \quad X(j\omega) = j \frac{d}{d\omega}\{W(j\omega)\}$$

$$= \begin{cases} j/2\pi & -2<\omega<0 \\ -j/2\pi & 0<\omega<2 \\ 0 & |\omega|>2 \end{cases}$$



(b) $$A = \int_{-\infty}^{\infty} t^2 \left(\frac{\sin t}{\pi t}\right)^4 dt$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

$$A = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

$$|X(j\omega)|^2 = \begin{cases} \frac{1}{2\pi} & -2 < \omega \leq 2 \\ 0 & |\omega| > 2 \end{cases}$$

$$A = \frac{1}{2\pi}\int_{-2}^{2} \left(\tfrac{1}{2\pi}\right)^2 d\omega$$

$$= \frac{1}{(2\pi)^3}\left[\omega \Big|_{-2}^{2}\right]$$

$$= \frac{1}{8\pi^3}(4)$$

$$= \underline{\underline{\frac{1}{2\pi^3}}} \quad \text{Ans}$$

4.26. a) (i) $x(t) = te^{-2t}u(t)$

$$X(j\omega) = \frac{1}{(2+j\omega)^2}$$

$\cancel{Y(j\omega)}$ $h(t) = e^{-4t}u(t)$

$$H(j\omega) = \frac{1}{4+j\omega}$$

$$Y(j\omega) = X(j\omega)\,H(j\omega)$$

$$Y(j\omega) = \frac{1}{(2+j\omega)^2}\left[\frac{1}{4+j\omega}\right]$$

$$= \frac{\frac{1}{4}}{4+j\omega} - \frac{1/4}{2+j\omega} + \frac{1/2}{(2+j\omega)^2}$$

$$y(t) = \frac{1}{4}e^{-4t}u(t) - \frac{1}{4}e^{-2t}u(t) + \frac{1}{2}e^{-2t}u(t)\,t$$

$$= \frac{1}{4}u(t)\left[e^{-4t} - e^{-2t} + 2te^{-2t}\right]$$

4.19) $$H(j\omega) = \frac{1}{j\omega+3}$$

$$h(t) = e^{-3t}u(t)$$

$$y(t) = e^{-3t}u(t) - e^{-4t}u(t)$$

$$Y(j\omega) = \frac{1}{j\omega+3} - \frac{1}{j\omega-4} = \frac{1}{(j\omega+3)(j\omega-4)}$$

$$X(j\omega) = \frac{Y(j\omega)}{H(j\omega)} = \frac{1}{j\omega-4}$$

$$x(t) = e^{-4t}u(t).$$

4.32) $h(t) = \frac{\sin[4(t-1)]}{\pi(t-1)}$

$$h(t) = h(t-1)$$

$$= \frac{\sin 4t}{\pi t}$$

$$H(j\omega) = \begin{cases} 1 & |\omega| < 4 \\ 0 & |\omega| > 4 \end{cases}$$

$$h(t-1) = \cancel{H(j\omega)} = e^{-j\omega} H(j\omega)$$

$$= \begin{cases} e^{-j\omega} & |\omega| < 4 \\ 0 & |\omega| > 4 \end{cases}$$

(a) $x_1(t) = \cos(6t + \pi/2)$

$$= \frac{1}{2}\left[e^{j(6t+\pi/2)} + e^{-j(6t+\pi/2)}\right]$$

$$= e^{j6t}\frac{e^{j\pi/2}}{2} + \frac{e^{-j\pi/2}}{2}e^{-j6t}$$

$$X_1(j\omega) = 2\pi\left[\frac{e^{j\pi/2}}{2}\cancel{2\pi}\,\delta(\omega-6) + \frac{e^{-j\pi/2}}{2}\delta(\omega+6)\right]$$

$$= \pi e^{j\pi/2}\delta(\omega-6) + \pi e^{-j\pi/2}\delta(\omega+6)$$

$$\therefore \quad Y_1(j\omega) = X_1(j\omega)\cdot H_1(j\omega)$$

$$= \underline{\underline{0}} \quad \underline{\underline{\text{Ans.}}}$$

(b) $$x_2(t) = \sum_{k=0}^{\infty} (1/2)^k \sin(3kt)$$

$$X_2(j\omega) = \sum_{k=0}^{\infty} (1/2)^k \mathcal{F}\left[\sin(3kt)\right]$$

$$= \sum_{k=0}^{\infty} (\tfrac{1}{2})^k \cdot \frac{2\pi}{2j}\left[\delta(\omega-3k) - \delta(\omega+3k)\right]$$

$$Y_2(j\omega) = X_2(j\omega)\, H_2(j\omega)$$

$$= \frac{\pi}{j}\sum_{k=0}^{1} (\tfrac{1}{2})^k \left[\delta(\omega-3k) - \delta(\omega+3k)\right] e^{-j\omega}$$

$$y_2(t) = \frac{1}{2}\sin(3t-1) \quad \underline{\underline{\text{Ans.}}}$$

(c) $$x_3(t) = \frac{\sin[4(t+1)]}{\pi(t+1)}$$

$$x(t) = \frac{\sin 4t}{\pi t}$$

$$X(j\omega) = \begin{cases} 1 & |\omega| < 4 \\ 0 & \text{elsewhere} \end{cases}$$

$$x(t) = x(t+1)$$

$$x(t+1) = e^{+j\omega} X(j\omega)$$

$$X_3(j\omega) = \begin{cases} e^{j\omega} & |\omega| < 4 \\ 0 & \text{elsewhere} \end{cases}$$

$$\cancel{H}\; Y(j\omega) = X_3(j\omega)\, H(j\omega)$$

$$= \begin{cases} e^{j\omega} e^{-j\omega} & |\omega| < 4 \\ 0 & \text{elsewhere} \end{cases}$$

$$= \begin{cases} 1 & |\omega| < 4 \\ 0 & \text{elsewhere} \end{cases}$$

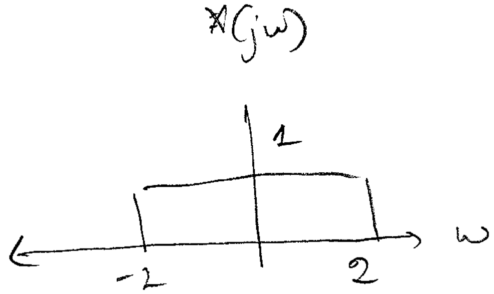
(d) $$x_4(t) = \left[\frac{\sin(2t)}{\pi t}\right]^2$$

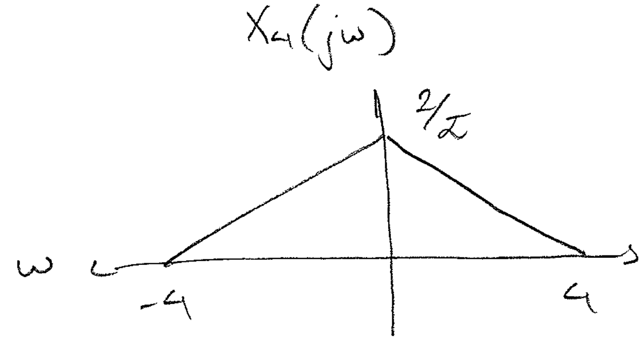
$$x(t) = \frac{\sin(2t)}{\pi t}$$

$$X(j\omega) = \begin{cases} 1 & |\omega| < 2 \\ 0 & \text{elsewhere} \end{cases}$$

$$x_4(t) = [x(t)]^2$$

$$X_4(j\omega) = \frac{1}{2\pi}\left[X(j\omega) * X(j\omega)\right]$$





$$\therefore \quad Y_4(j\omega) = X_4(j\omega)\, H(j\omega)$$

$$= X_4(j\omega)\, e^{-j\omega}$$

$$y_4(t) = x_4(t-1)$$

$$= \left[\frac{\sin[2(t-1)]}{\pi(t-1)} \right]^2 \quad \text{Ans:}$$

4.33.) $\frac{d^2}{dt^2} y(t) + 6 \frac{d}{dt} y(t) + 8y(t) = 2x(t)$

(a) $Y(j\omega) \left[(j\omega)^2 + 6j\omega + 8 \right] = 2X(j\omega)$

$$\frac{Y(j\omega)}{X(j\omega)} = H(j\omega) = \frac{2}{-\omega^2 + 6j\omega + 8}$$

$$= \frac{1}{j\omega + 2} - \frac{1}{j\omega + 4}$$

$$h(t) = e^{-2t} u(t) - e^{-4t} u(t)$$

(b) $x(t) = te^{-2t}u(t)$

$$X(j\omega) = \frac{1}{(2+j\omega)^2}$$

$$Y(j\omega) = \frac{1}{(2+j\omega)^2}\left[\frac{2}{(j\omega+2)(j\omega+4)}\right]$$

$$= \frac{2}{(j\omega+2)^3(j\omega+4)}$$

$$= \frac{1/4}{j\omega+2} - \frac{1/2}{(j\omega+2)^2} + \frac{1}{(j\omega+2)^3} - \frac{1/4}{j\omega+4}$$

$$y(t) = \tfrac{1}{4}e^{-2t}u(t) - \tfrac{1}{2}e^{-2t}u(t) + e^{-2t}u(t) - \tfrac{1}{4}e^{-4t}u(t).$$

(c) $\frac{d^2}{dt^2}y(t) + \sqrt{2}\,\frac{d}{dt}y(t) + y(t) = 2\frac{d^2}{dt^2}x(t) - 2x(t)$

$$Y(j\omega)\left[(j\omega)^2 + \sqrt{2}j\omega + 1\right] = X(j\omega)\left[2(j\omega)^2 - 2\right]$$

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{2(j\omega)^2 - 2}{(j\omega)^2 + \sqrt{2}j\omega + 1} = \frac{-2(\omega^2+1)}{-\omega^2 + \sqrt{2}j\omega + 1}$$

$$= 2 + \frac{-\sqrt{2} - 2\sqrt{2}j}{j\omega - \frac{-\sqrt{2}+j\sqrt{2}}{2}} + \frac{-\sqrt{2} + 2\sqrt{2}j}{j\omega - \frac{-\sqrt{2}-j\sqrt{2}}{2}}$$

$$\therefore \quad h(t) = 2\delta(t) - \sqrt{2}(1+2j)\, e^{-(1+j)t/\sqrt{2}}\, u(t)$$
$$- \sqrt{2}(1-2j)\, e^{-(1-j)t/\sqrt{2}}\, u(t).$$

Ans.

4.34) (a) $$H(j\omega) = \frac{j\omega + 4}{6 + (j\omega)^2 + 5j\omega}$$

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega + 4}{(j\omega)^2 + 5j\omega + 6}$$

$$Y(j\omega)\left[(j\omega)^2 + 5j\omega + 6\right] = X(j\omega)\left[j\omega + 4\right]$$

$$\frac{d^2}{dt^2} y(t) + 5\frac{d}{dt} y(t) + 6y(t) = \frac{d}{dt} x(t) + 4x(t)$$

(b) $$H(j\omega) = \frac{2}{2+j\omega} - \frac{1}{3+j\omega}$$ {Using partial fraction}

$$h(t) = 2e^{-2t}u(t) - e^{-3t}u(t).$$

(c) $$x(t) = e^{-4t}u(t) - te^{-4t}u(t)$$

$$X(j\omega) = \frac{1}{4+j\omega} - \frac{1}{(4+j\omega)^2}$$

$$Y(j\omega) = \frac{1}{(4+j\omega)(2+j\omega)}$$

Using partial fractions.

$$y(t) = \frac{1}{2}e^{-2t}u(t) - \frac{1}{2}e^{-4t}u(t)$$