M. B. P. Problem 1.8 p. 20

1.8 Two random variables, X_1 and X_2, are produced in a digital communication receiver. An error occurs if $X_1 > X_2$; otherwise, the receiver output is correct. The random variables X_1 and X_2 are independent, Gaussian random variables, with means μ_1 and μ_2 and standard deviations σ_1 and σ_2, respectively. Find the probability of error for this receiver. First express your answer in terms of the function Φ and the parameters μ_1, μ_2, σ_1, and σ_2. Convert your answer so that it is in terms of the function Q. (*Hint*: perhaps the best approach is to let $X = X_1 - X_2$ so that X is a Gaussian random variable and $P(X > 0)$ is the probability of error for the communication receiver.)

MBP 1.8

$X_1 \sim N(\mu_1, \sigma_1^2)$ $\quad X_1 \perp\!\!\!\perp X_2$

$X_2 \sim N(\mu_2, \sigma_2^2)$

$E =$ event an error occurs

$= \{X_1 > X_2\}$

Find $P(E)$.

Let $X = X_1 - X_2 \Longrightarrow E = \{X > 0\}$ and note that

$X \sim$ Gaussian

since linear combinations of Gaussian rvs are Gaussian.

$EX = EX_1 - EX_2 = \mu_1 - \mu_2$

$Var(X) = Var(X_1) + Var(X_2)$ (uses independence)

$= \sigma_1^2 + \sigma_2^2$

$$P(X > 0) = P\left(\frac{X - \mu_1 + \mu_2}{\sqrt{\sigma_1^2 + \sigma_2^2}} > \frac{-\mu_1 + \mu_2}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right)$$

$$= Q\left(\frac{\mu_2 - \mu_1}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right).$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy \qquad -\infty < x < \infty$$

$$\Phi(-x) = 1 - \Phi(x)$$

$$\triangleq Q(x)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-y^2/2}\, dy$$

M. B. P. Problem 2.7 p 64

2.7 Suppose X is a random variable that is uniformly distributed on $[0, 1]$. The random process $Y(t)$, $t > 0$, is defined by $Y(t) = \exp\{-Xt\}$. Find the one-dimensional distribution function $F_{Y,1}(u; t)$ for the random process $Y(t)$. Find the one-dimensional density function $f_{Y,1}(u; t)$.

MBP 2.7

$X \sim$ unif. $[0,1]$

$Y(t) = e^{-Xt} \quad t > 0$

↳ the sample functions are all decaying exponentials.

- $\lim_{t \to 0^+} Y(t) = 1 \quad \forall$ samp. functs.
- $\lim_{t \to \infty} Y(t) = 0$ for almost all sample functs.

The one dim. dist is defined as

$$F_{Y,1}(u;t) = P(Y(t) \leq u)$$

$$= P(e^{-Xt} \leq u)$$

Probably should look at cases. Note that

$$0 < e^{-Xt} \leq 1$$

Hence

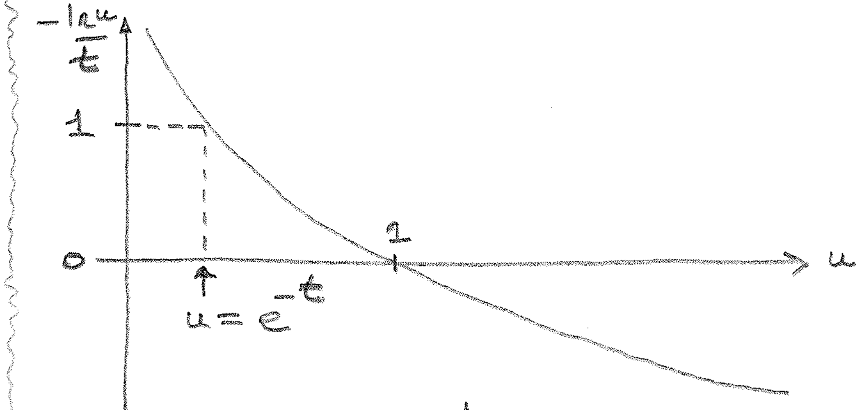
If $u \leq 0 \implies F_{Y,1}(u;t) \equiv 0$

If $u \geq 1 \implies F_{Y,1}(u;t) \equiv 1$

Consider $0 < u < 1$ and $t > 0$

$$e^{-Xt} \leq u \iff X \geq -\frac{\ln u}{t}$$

and remember that $0 \leq X \leq 1$ since it is unif. on that interval



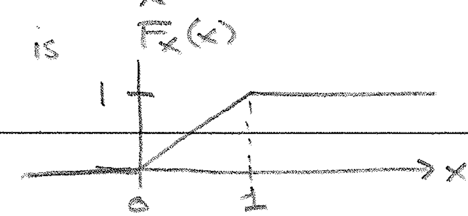
Thus if $0 < u < e^{-t}$ then $P(X \geq -\frac{\ln u}{t}) = 0$

and if $e^{-t} < u < 1$

then
$$P(X \geq -\tfrac{\ln u}{t}) = P(X > -\tfrac{\ln u}{t})$$
$$= 1 - F_X(-\tfrac{\ln u}{t}).$$

where F_X is cdf of unif $[0,1]$ r.v.

That is



That is $F_X(x) = x$ for $0 \le x \le 1$.

Putting the pieces together

$$F_{Y,1}(u;t) = \begin{cases} 0 & u < e^{-t} \\ 1 + \frac{\ln u}{t} & e^{-t} < u < 1 \\ 1 & u > 1 \end{cases}$$

Can check but this cdf is continuous (no jumps) over $u \in \mathbb{R}$ and $t > 0$.

The one dimensional pdf is the deriv. of above wrt u.

$$f_{Y,1}(u;t) = \frac{\partial}{\partial u} F_{Y,1}(u;t)$$

$$= \begin{cases} 0 & u < e^{-t} \\ 1/ut & e^{-t} < u < 1 \\ 0 & u > 1 \end{cases}$$

The pdf is not continuous as it jumps at $u = e^{-t}, 1$

M. B. P. Problem 2.10 p65

2.10 The wide-sense stationary random process $X(t)$, $t \in \mathbb{R}$, is a Gaussian random process with zero mean and autocorrelation function $R_X(\tau)$. Find the one-dimensional distribution function for the random process $Y(t) = \Phi\left[X(t)/\sqrt{R_X(0)}\right]$ for $-\infty < t < \infty$. Find the one-dimensional density function for $Y(t)$. Is the process $Y(t)$ stationary?

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MBP 2.10

$X(t)$ Gaussian 0 mean, WSS, $R_X(\tau)$

$$Y(t) \triangleq \Phi\left[\underbrace{X(t)/\sqrt{R_X(0)}}\right] \qquad t \in \mathbb{R}$$

↳ this process is Gauss. 0 mean, WSS, with $\frac{R_X(\tau)}{R_X(0)}$ for auto corr.

$F_{Y,1}(u;t) = P(Y(t) \le u)$. But note due to special def. of r.p. $Y(t)$ that

$$0 \le Y(t) \le 1$$

with prob. one for all t. Therefore

$$F_{Y,1}(u;t) = 0 \text{ for } u<0, \forall t$$
$$= 1 \text{ for } u>1, \forall t$$

Consider $0<u<1$. Then the following are equiv. events

$$\Phi\left(\frac{X(t)}{\sqrt{R_X(0)}}\right) \le u \iff \frac{X(t)}{\sqrt{R_X(0)}} \le \Phi^{-1}(u)$$

The prob. of this event is

$$F_{Y,1}(u;t) = \Phi(\Phi^{-1}(u)) = u$$

$$\therefore F_{Y,1}(u;t) = \begin{cases} 0 & u<0 \\ u & 0<u<1 \\ 1 & u>1 \end{cases}$$

which is the unif $[0,1]$ dist.

Clearly the one-dim. pdf is

$$f_{Y,1}(u;t) = \begin{cases} 0 & u<0 \\ 1 & 0<u<1 \\ 0 & u>1. \end{cases}$$

By definition the one-dim. dist. are stationary. To claim the proc. is stationary we have to show

$$F_{Y,n}(y_1 y_2 \cdots y_n; t_1 \ldots t_n)$$
$$= F_{Y,n}(y_1 y_2 \cdots y_n; t_1+t_0, t_2+t_0 \ldots t_n+t_0)$$

$\forall t_0$

It is actually true that a memoryless transf. of a SSS process is SSS.

Thus true here since Gaussian + WSS $\Longrightarrow$ SSS.

M. B. P. Problem 2.16 p 65

2.16 Suppose $X(t)$ and $Y(t)$ are zero-mean, wide-sense stationary, continuous-time random processes. If $X(t)$ and $Y(t)$ are independent, find the autocorrelation function for $Z(t)$ in terms of the autocorrelation functions for $X(t)$ and $Y(t)$ in each of the cases that follow. In each case, determine whether the random process $Z(t)$ is wide-sense stationary.

(a) $Z(t) = c\,X(t)\,Y(t) + d$, where c and d are deterministic constants.

(b) $Z(t) = X(t)\cos(\omega_0 t) + Y(t)\sin(\omega_0 t)$.

MBP 2.16 $X(t), Y(t)$ 0 mean, WSS, indep of each other.

(a) $Z(t) = c X(t) Y(t) + d$

$EZ(t) = d$ (does not dep. on t)

$$E\{Z(t)Z(s)\} = E\{(cX(t)Y(t)+d)(cX(s)Y(s)+d)\}$$

$$= c^2 E\{X(t)X(s)\}E\{Y(t)Y(s)\} + cd\, E\{\cancel{X(t)}\}E\{\cancel{Y(t)}\} + cd\, E\{\cancel{X(s)}\}E\{\cancel{Y(s)}\} + d^2$$

0 0 0 0

$$= c^2 R_X(t-s) R_Y(t-s) + d^2 \longrightarrow$$ funct. of $t-s$ only.

$\therefore$ WSS

(b) $Z(t) = X(t)\cos\omega_0 t + Y(t)\sin\omega_0 t$

$EZ(t) = 0$

$$E\{Z(t)Z(s)\} =$$

$$E\{(X(t)\cos\omega_0 t + Y(t)\sin\omega_0 t)(X(s)\cos\omega_0 s + Y(s)\sin\omega_0 s)\}$$

$$= R_X(t-s)\underbrace{\cos\omega_0 t\cos\omega_0 s}_{\frac{\cos\omega_0(t-s)+\cos\omega_0(t+s)}{2}} + R_Y(t-s)\underbrace{\sin\omega_0 t\sin\omega_0 s}_{\frac{\cos\omega_0(t-s)-\cos\omega_0(t+s)}{2}}$$

$$= \frac{1}{2}\cos\omega_0(t-s)\left[R_X(t-s) + R_Y(t-s)\right]$$

$$+ \frac{1}{2}\cos\omega_0(t+s)\left[R_X(t-s) - R_Y(t-s)\right]$$

In general depends also on $t+s$; hence not WSS in gen.

<u>Note</u> Is WSS if have additional assump. that

$$R_X(\tau) = R_Y(\tau).$$

Problem 7. You are trapped in a funhouse at a place containing three doors. The first door leads to the outside after 5 minutes of travel. The second door leads to a twisty tunnel that comes back to your starting spot after 20 minutes. The third door is similarly twisty turny and leads back to the same spot after 30 minutes. If you are equally likely to choose any one of the three doors whenever presented with the choice (i.e., you can't remember which door you just tried assuming you come around again), what is the expected length of time until you escape.

Problem 7

$D =$ door chosen $(= 1,2,3)$ $\quad P\{D=i\} = 1/3 \quad i=1,2,3$

$T =$ time until escape

$$E\{T\} = E\{T|D=1\}P\{D=1\} + E\{T|D=2\}P\{D=2\} + E\{T|D=3\}P\{D=3\}$$

$$= \frac{1}{3}\Big[\underbrace{E\{T|D=1\}}_{5} + \underbrace{E\{T|D=2\}}_{20+E\{T\}} + \underbrace{E\{T|D=3\}}_{30+E\{T\}}\Big]$$

Because when you come back around the same problem still exists since you cannot remember which door you selected before.

$$\therefore E\{T\} = \frac{1}{3}\left[5 + 20 + E\{T\} + 30 + E\{T\}\right]$$

$$3E\{T\} = 55 + 2E\{T\} \Longrightarrow E\{T\} = 55 \text{ minutes.}$$

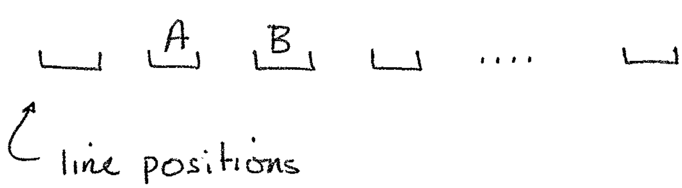
Problem 12. If N people, including A and B, are randomly arranged in a line, what is the probability that A and B are next to each other? What would the probability be if the people were randomly arranged in a circle?

Problem 12

N people (distinguishable) randomly arranged in a line. There are $N!$ ways to order the people in the line. Saying the people are randomly arranged amounts to assuming that each of the $N!$ orderings are equally likely:

$$P\left\{\begin{matrix}\text{any particular ordering} \\ \text{of the people}\end{matrix}\right\} = 1/N!$$

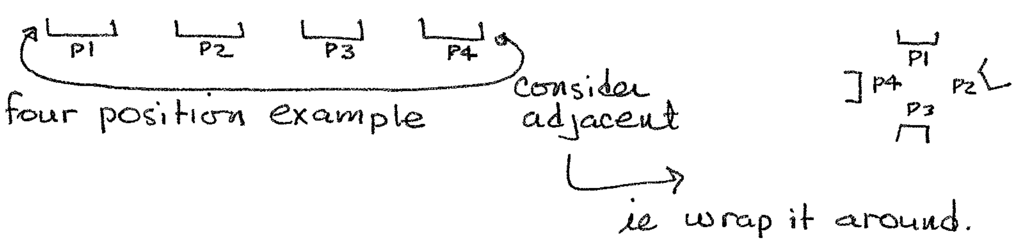
To compute the prob. that 2 of the people (named A and B) are next to each other we need to find the number of ways that this can be done. A particular way:



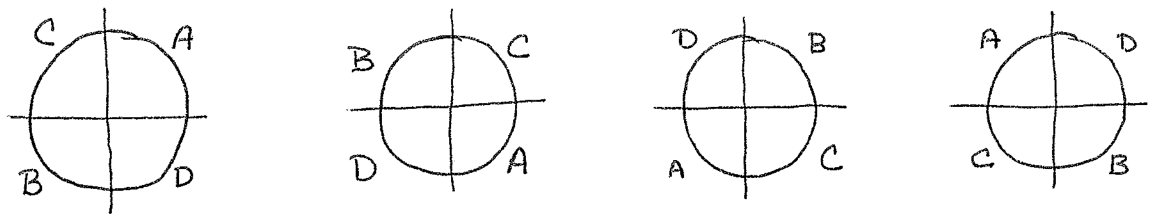
If we treat AB as a single entity see that there are $(N-1)!$ orderings of the people in which A, and B appear together in the order AB. Similarly, there are $(N-1)!$ orderings of the people in which A and B appear together in the order BA. In sum there are $(N-1)!\,2!$ ordering where A and B are together.

$$P\left\{\begin{matrix}\text{A, B together in a} \\ \text{random linear ordering}\end{matrix}\right\} = \frac{(N-1)!\,2!}{N!} = \frac{2}{N}$$

N people (distinguishable) randomly arranged in a circle We can get at this problem from the results of the line case by recognizing that the line can be made into a circle by saying that the first position and the last position are adjacent:



If we could solve the problem of randomly arranging N objects in a circle the result for this problem could be quickly found. If we only care about the order in the circle we consider all rotations as being the same. For example



(here $N=4$) the four seating arrangements above would be considered identical if it is only order around the circle that is of interest. Since there are N equivalent rotations for each ordering:

$$\frac{N!}{N} = \text{\# orderings of } N \text{ people in a circle.}$$

To finish the problem consider A, B as a group. Then there are

$$\frac{(N-1)!\,2!}{N-1}$$

orderings where A, B are together around circle. Thus

$$P\left\{\begin{array}{c}\text{A, B together in}\\ \text{a random circular}\\ \text{ordering}\end{array}\right\} = \frac{(N-1)!\,2!}{N-1}\,\frac{N}{N!} = \frac{2}{N-1}$$

Problem 22. A random process is composed of sample functions of the form

$$X(t) = Ae^{-\alpha t}u(t - \Delta)$$

where A is a Gaussian random variable with mean zero and variance σ_A^2, α is a constant, and Δ is uniformly distributed in the interval $(0, T)$ and independent of A.

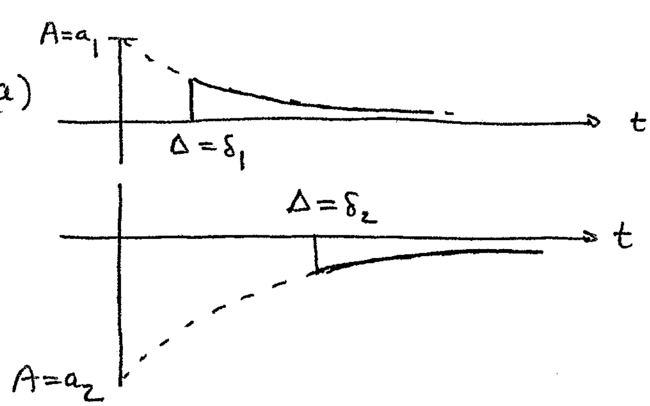
(a) Sketch several sample functions.

(b) Is this process WSS?

Problem 22

$$X(t) = A e^{-at} u(t-\Delta) \qquad \begin{array}{l} A \sim \mathcal{N}(0, \sigma_A^2) \\ \Delta \sim \mathcal{U}(0, T) \end{array} \qquad A \perp\!\!\!\perp \Delta$$

(a)

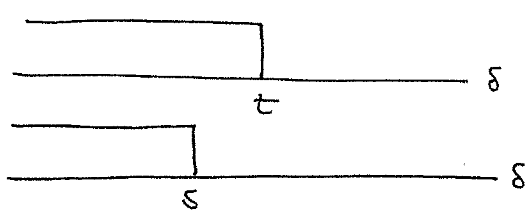


(b) $E\{X(t)\} = 0$

$$E\{X(t)X(s)\} = E\left\{A^2 e^{-\alpha(t+s)} u(t-\Delta)u(s-\Delta)\right\}$$

$$= \sigma_A^2 e^{-\alpha(t+s)} \, E\left\{u(t-\Delta)u(s-\Delta)\right\}$$

$$= \frac{1}{T}\int_0^T u(t-\delta)u(s-\delta)\,d\delta = \begin{cases} 0 & \text{if } \min(s,t) \le 0 \\ 1 & \text{if } \min(s,t) \ge T \\ \frac{\min(s,t)}{T} & 0 < \min(s,t) < T \end{cases}$$



Clearly depends on <u>both</u> t and s. Not WSS.

Problem 23. Consider the WSS noise waveforms $N_k(t)$, $1 \le k \le N$, where $\mathrm{E}\{N_k(t)\} = 0$ for $1 \le k \le N$ and

$$\mathrm{E}\{N_j(t)N_k(t)\} = \begin{cases} \sigma^2 & j = k \\ \sigma^2/2 & |j-k| = 1 \\ 0 & |j-k| > 1 \end{cases} .$$

(a) Calculate the second moment of

$$N(t) = \sum_{k=1}^{N} N_k(t).$$

What is the variance of $N(t)$?

(b) Assuming that each of the $N_k(t)$ has a Gaussian pdf, write down the pdf of $N(t)$.

Problem 23

$E\{N(t)\} = 0 \Rightarrow$ second moment = variance

$$E\{N^2(t)\} = E\left\{\sum_{j=1}^{N}\sum_{i=1}^{N} N_j(t)\,N_i(t)\right\} = \sum_{j=1}^{N}\sum_{i=1}^{N} E\{N_j(t)\,N_i(t)\}$$

$$= \sum_{j=1}^{N} E\{N_j^2(t)\} + \sum_{j=2}^{N} E\{N_j(t)\,N_{j-1}(t)\} + \sum_{j=1}^{N-1} E\{N_j(t)\,N_{j+1}(t)\}$$

$$= N\sigma^2 + (N-1)\sigma^2/2 + (N-1)\sigma^2/2 = (2N-1)\sigma^2$$

$$f_{N(t)}(n_t) = \frac{1}{\sqrt{2\pi}\sqrt{2N-1}\,\sigma}\, e^{-n_t^2/2(2N-1)\sigma^2}$$