ECE 302

Spring 2013

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Practice Final Examination

Instructions:

1. Wait for the "BEGIN" signal before opening this booklet. In the meantime, read the instructions below and fill out the requested info.
2. You have 120 minutes to complete this exam. **When the end of the exam is announced, you must stop writing immediately**. Anyone caught writing after the exam is over will get a grade of zero on this exam.
3. At the end of this document is a table of formulas and some scratch paper. Do **not** tear out any of these pages.
4. This is a closed book exam. The only personal items allowed are pens/pencils, erasers, your Purdue ID and something to drink. Anything else is strictly forbidden (esp. anything with wireless communication capability, including watches).
5. You must keep your eyes on your exam at all times. Looking around is strictly forbidden.
6. Please leave your Purdue ID out so the proctors may check your identity.

Name:____________________________________

Email:____________________________________

Signature:________________________________

Itemized Scores

Problem 1:	Problem 6:
Problem 2:	Problem 7:
Problem 3:	
Problem 4:	
Problem 5:	
Total:	

(15 pts) **1.** Let A be the event: it is sunny today. The B be the event: Joe ate cereal for breakfast today. Assume that A and B are independent events, and that P(A)=0.5 and P(B)=0.3. What is the probability that only one of these events (either A or B but not both) occurs? Carefully derive the numerical answer, justifying every step along the way using the axioms of probability.

(30 pts) **2.** a) Obtain the characteristic function $M_X(s) = E(e^{sX})$ of an exponential random variable X. (Recall that the pdf of an exponential random variable is given by $f_X(x) = \lambda e^{-\lambda x}$, for $x \geq 0$.)

b) What, if any, is the relationship between the characteristic function of a 1D random variable X and the moments of X? Explain.

(20 pts) **3.**

a) Let Z be the number of success in n (independent) Bernoulli trials. What is the pmf of Z? (No justification needed, but make sure to clearly write your answer.)

b) Let W be the position after n steps of a one-dimensional random walk. Assume that each step is randomly (independently) determined according to the random variable $Y = 2X - 1$ where X is a Bernoulli random variable. What is the pmf $p_W(k)$ of W? (No justification needed, but make sure to clearly write your answer.)

(15 pts) **4.** Let $S = (S_t)_{t \in R}$ be a Poisson process. Recall that the pmf for the random variable S_t is $p_{S_t}(k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$. Obtain an approximate expression for the probability of obtaining 1 success in the time interval $[0, t]$ for t small. (You must show the intermediate steps you used to obtain your answer in order to get full credit. You are allowed to use the facts contained in the table without justification.)

(15 pts) **5.** Let T be the time when the first event occurs in a Poisson random process. Obtain the pdf of T.

(20 pts) **6.** Let X be a wide sense stationary continuous-time random process with autocorrelation function $\bar{R}_X(t)$. Find the power spectral density $SP_Y(f)$ of a random process Y with autocorrelation function $\bar{R}_X(t)\cos(2\pi f_0 t)$. Justify your answer. (You may use any fact contained in the table to justify your answer.)

(20 pts) **7.** Let $X = (X_t)_{t \in R}$ be a wide sense stationary random process. Assume that X_t is differentiable. Define

$$Y(t) = \frac{dX_t}{dt}.$$

What is the mean $m_Y(t)$ and the autocorrelation function $R_Y(t_1, t_2)$ of Y? (Justify your answer using the facts contained in the table.)

Table

Power series

Function	Power Series	Region of convergence	
$\frac{1}{1-x}$	$\sum_{k=0}^{\infty} x^k$	$\lvert x\rvert < 1$	(1)
e^x	$\sum_{k=0}^{\infty} \frac{x^k}{k!}$	any real or complex x	(2)
$(a+b)^n$	$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$	any real or complex a, b	(3)

CT Signal Energy and Power

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \tag{4}$$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt \tag{5}$$

Fourier Series of CT Periodic Signals with period T

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t} \tag{6}$$

$$a_k = \frac{1}{T} \int_0^T x(t) e^{-jk\left(\frac{2\pi}{T}\right)t} dt \tag{7}$$

Power Spectral Density of a Random Process $X = (X_t)_{t\in R}$

$$SP_X(f) = \int_{-\infty}^{\infty} X_t e^{-j2\pi f t} dt \tag{8}$$

$$= \lim_{T\to\infty} \frac{1}{T} E\left(\left|\int_{-\frac{T}{2}}^{\frac{T}{2}} X_t e^{-j2\pi f t} dt\right|^2\right) \tag{9}$$

Processing of a w.s.s. Random Process $X = (X_t)_{t \in R}$ with an LTI system

$$E(Y_t) = E(X_t)H(0) \tag{10}$$
$$SP_Y(f) = |H(f)|^2 \, SP_X(f) \tag{11}$$

CT Fourier Transform

$$\text{F.T. : } \mathcal{X}(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \tag{12}$$
$$\text{Inverse F.T.: } x(t) = \int_{-\infty}^{\infty} \mathcal{X}(f) e^{j2\pi f t} df \tag{13}$$

Properties of CT Fourier Transform

Let $x(t)$ be a continuous-time signal and denote by $\mathcal{X}(f)$ its Fourier transform. Let $y(t)$ be another continuous-time signal and denote by $\mathcal{Y}(f)$ its Fourier transform.

	Signal	*FT*	
Linearity:	$ax(t) + by(t)$	$a\mathcal{X}(f) + b\mathcal{Y}(f)$	(14)
Time Shifting:	$x(t - t_0)$	$e^{-j\omega t_0}\mathcal{X}(f)$	(15)
Frequency Shifting:	$e^{j2\pi f_0 t}x(t)$	$\mathcal{X}(f - f_0)$	(16)
Duality	$\mathcal{X}(t)$	$x(-f)$	(17)
Time and Frequency Scaling:	$x(\frac{t}{a})$	$\|a\|X\,(af)$	(18)
Multiplication:	$x(t)y(t)$	$\mathcal{X}(f) * \mathcal{Y}(f)$	(19)
Convolution:	$x(t) * y(t)$	$\mathcal{X}(f)\mathcal{Y}(f)$	(20)
Transform of periodic signals	$\mathrm{rep}_T[x(t)]$	$\frac{1}{T}\mathrm{comb}_{\frac{1}{T}}[\mathcal{X}(f)]$	(21)
Transform of sampled signals	$\mathrm{comb}_T[x(t)]$	$\mathrm{rep}_{\frac{1}{T}}[\mathcal{X}(f)]$	(22)
			(23)

Some CT Fourier Transform Pairs

$$e^{j2\pi f_0 t} \xrightarrow{\mathcal{F}} \delta(f - f_0) \tag{24}$$
$$1 \xrightarrow{\mathcal{F}} \delta(f) \tag{25}$$
$$\frac{\sin(2\pi f_0 t)}{\pi t} \xrightarrow{\mathcal{F}} u(f + f_0) - u(f - f0) = \mathrm{rect}\left(\frac{f}{2f_0}\right) \tag{26}$$
$$\delta(t) \xrightarrow{\mathcal{F}} 1 \tag{27}$$
$$\sum_{n=-\infty}^{\infty} \delta(t - nT) \xrightarrow{\mathcal{F}} \frac{1}{T}\sum_{k=-\infty}^{\infty} \delta(f - \frac{k}{T}) \tag{28}$$

Laplace Transform

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{29}$$

Properties of Laplace Transform

Let $x(t)$, $x_1(t)$ and $x_2(t)$ be three CT signals and denote by $X(s)$, $X_1(s)$ and $X_2(s)$ their respective Laplace transform. Let R be the ROC of $X(s)$, let R_1 be the ROC of $X_1(z)$ and let R_2 be the ROC of $X_2(s)$.

	Signal	L.T.	ROC	
Linearity:	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$	(30)
Time Shifting:	$x(t - t_0)$	$e^{-st_0}X(s)$	R	(31)
Shifting in s:	$e^{s_0 t}x(t)$	$X(s - s_0)$	$R + s_0$	(32)
Conjugation:	$x^*(t)$	$X^*(s^*)$	R	(33)
Time Scaling:	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	aR	(34)
Convolution:	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$	(35)
Differentiation in Time:	$\frac{d}{dt}x(t)$	$sX(s)$	At least R	(36)
Differentiation in s:	$-tx(t)$	$\frac{dX(s)}{ds}$	R	(37)
Integration :	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	At least $R \cap \mathcal{R}e\{s\} > 0$	(38)

Some Laplace Transform Pairs

Signal	*LT*	*ROC*	
$\delta(t)$	1	all s	(39)
$u(t)$	$\frac{1}{s}$	$\mathcal{R}e\{s\} > 0$	(40)
$u(t)\cos(\omega_0 t)$	$\frac{s}{s^2 + \omega_0^2}$	$\mathcal{R}e\{s\} > 0$	(41)
$e^{-\alpha t}u(t)$	$\frac{1}{s + \alpha}$	$\mathcal{R}e\{s\} > -\alpha$	(42)
$-e^{-\alpha t}u(-t)$	$\frac{1}{s + \alpha}$	$\mathcal{R}e\{s\} < -\alpha$	(43)

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