7/17 ①

Convolution of Two Gaussian Pulses.

$$\frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_1^2}} * \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_2^2}}$$

$$= \frac{1}{\sigma_3\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_3^2}} \quad \text{where } \sigma_3^2 = \sigma_1^2 + \sigma_2^2$$

$(\sigma_i : \text{stdev})$

$$\left(\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}} dt = 1 \right)$$

$$\boxed{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}} \xleftrightarrow{\mathcal{F}} e^{-\frac{w^2\sigma^2}{2}}}$$

- We know from Time-Invariance

$$\frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{(t-t_1)^2}{2\sigma_1^2}} * \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{(t-t_2)^2}{2\sigma_2^2}}$$

$$= \frac{1}{\sigma_3\sqrt{2\pi}} e^{-\frac{(t-(t_1+t_2))^2}{2\sigma_3^2}}, \quad \sigma_3^2 = \sigma_1^2 + \sigma_2^2$$

<u>Variances add</u>

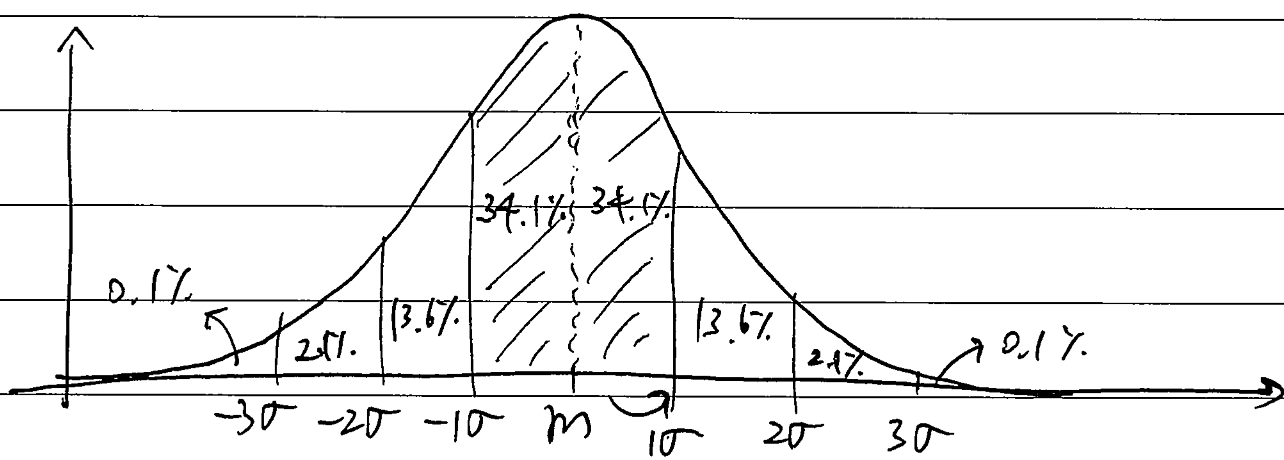
Fourier Transform of Gaussian Pulse

~~Preliminary Result~~ :

- $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}} dx = 1$, regardless of m. (also holds if m is purely imag)

- $\displaystyle\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx = \sqrt{2\pi}\,\sigma$

where independent var. x could be time (t) or freq. (ω).



m : mean
σ : stdev
σ^2 : variance

→ over 99% of the area is within $-3\sigma < x-m < 3\sigma$
($m-3\sigma < x < m+3\sigma$)

* stdev : standard deviation

- Show: $x(t) = e^{-\frac{t^2}{2}} \xleftrightarrow{\mathcal{F}} X(\omega) = e^{-\frac{\omega^2}{2}}\sqrt{2\pi}$

$$X(\omega) = \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}(t^2+2j\omega - \omega^2)}\,dt\; e^{-\frac{\omega^2}{2}}$$

$$= \underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{2}(t+j\omega)^2}\,dt}_{=\sqrt{2\pi}}\; e^{-\frac{\omega^2}{2}}$$

$$= \sqrt{2\pi}\, e^{-\frac{\omega^2}{2}}$$

$(\sigma = 1,\ m = j\omega)$ from previous page.

- (Recall) $x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$, $a = \frac{1}{\sigma}$

Thus, $e^{-\frac{t^2}{2\sigma^2}} \xleftrightarrow{\mathcal{F}} \sqrt{2\pi}\,\sigma\, e^{-\frac{\omega^2\sigma^2}{2}}$

Rearranging, $\boxed{e^{-\frac{t^2}{2\sigma^2}} \xleftrightarrow{\mathcal{F}} \sqrt{2\pi}\,\sigma\, e^{-\frac{\omega^2}{2/\sigma^2}}}$

(stdev $= \sigma$) (stdev $= \frac{1}{\sigma}$)

- What if square Gaussian or multiply two Gaussians?

$$e^{-\frac{t^2}{2\sigma_1^2}} e^{-\frac{t^2}{2\sigma_2^2}} = e^{-\frac{t^2}{2}\left(\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right)} = e^{-\frac{t^2}{2\sigma^2}}$$

$$\left(\sigma^2 = \frac{1}{\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}} \right)$$: harmonic mean.

→ product of 2 Gaussians (zero means) is another Gaussian with zero mean

- Convolution of Two Gaussians.

$$e^{-\frac{t^2}{2\sigma_1^2}} * e^{-\frac{t^2}{2\sigma_2^2}} \xleftrightarrow{\mathcal{F}} \sqrt{2\pi}\,\sigma_1 \sqrt{2\pi}\,\sigma_2\, e^{-\frac{\omega^2\sigma_1^2}{2}} e^{-\frac{\omega^2\sigma_2^2}{2}}$$

$$= 2\pi\sigma_1\sigma_2 \frac{\sqrt{2\pi}\sqrt{\sigma_1^2+\sigma_2^2}}{\sqrt{2\pi}\sqrt{\sigma_1^2+\sigma_2^2}} e^{-\frac{\omega^2}{2}(\sigma_1^2+\sigma_2^2)}$$

Going back to time-domain

$$\sqrt{2\pi}\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}} e^{-\frac{t^2}{2(\sigma_1^2+\sigma_2^2)}}$$

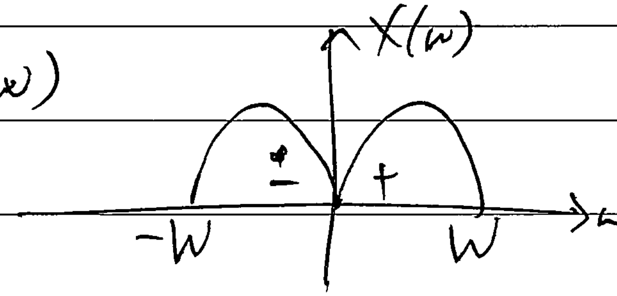
$$= \frac{\sqrt{2\pi}}{\sqrt{\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}}} e^{-\frac{t^2}{2(\sigma_1^2+\sigma_2^2)}}$$

: conv. of two Gaussians is a Gaussian with the sum of the variances.

Single Sideband Modulation.

Removing the negative freq. sideband in the transmitted spectrum to save bandwidth.

- Consider $x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$

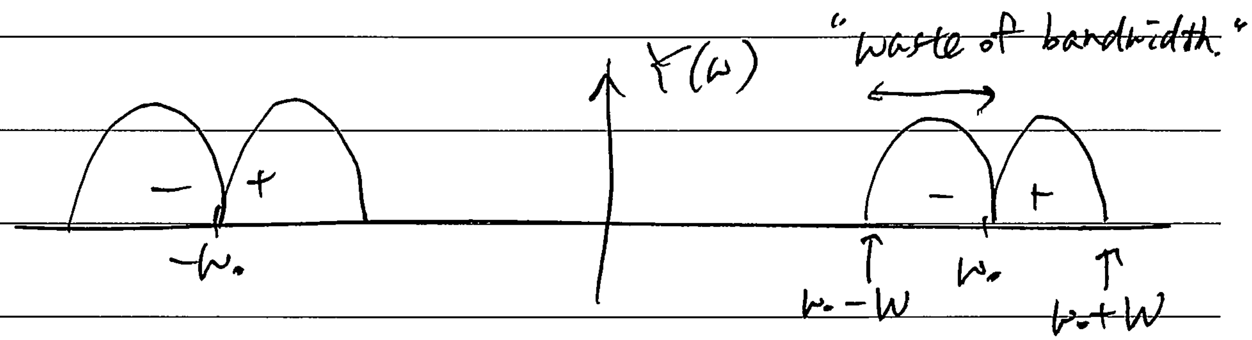


- Recall : if $x(t)$: real-valued
  then $X(-\omega) = X^*(\omega)$

  → $|X(\omega)| = |X(\omega)|$: magnitude is symmetric about $\omega = 0$

  → can imply negative freq. content.

- $y(t) = 2x(t)\cos(\omega_0 t) \xleftrightarrow{\mathcal{F}} X(\omega-\omega_0) + X(\omega+\omega_0) = Y(\omega)$



- To remove the negative freq. sideband, consider the following steps

1) Form $\hat{x}(t)$ by passing $x(t)$ thru an LTI system whose freq. response is

$$\underbrace{h_{HT}(t)}_{\text{(impulse response)}} = \frac{1}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow} \underbrace{H_{HT}(\omega)}_{\text{(freq. response)}} = \begin{cases} j, & \omega < 0 \\ -j, & \omega > 0 \end{cases} = j u(-\omega) - j u(\omega)$$

↳ this is called Hilbert Transformer.

$$x(t) \longrightarrow \boxed{H_{HT}(\omega)} \longrightarrow \hat{x}(t)$$

$$\hat{x}(t) = x(t) * h_{HT}(t) \underset{\mathcal{F}}{\longleftrightarrow} \hat{X}(\omega) = H_{HT}(\omega) X(\omega)$$

$$= \begin{cases} j X(\omega), & \omega < 0 \\ -j X(\omega), & \omega > 0 \end{cases}$$

2) Form complex-valued signal

$$\tilde{x}(t) = x(t) + j \hat{x}(t)$$

examine what happens in freq-domain

$$\tilde{X}(\omega) = X(\omega) + j\hat{X}(\omega)$$

$$= \begin{cases} X(\omega) + j\left(jX(\omega)\right), & \omega < 0 \\ X(\omega) + j\left(-jX(\omega)\right), & \omega > 0 \end{cases}$$

$$= \begin{cases} 0, & \omega < 0 \\ 2X(\omega), & \omega > 0 \end{cases}$$

$\Rightarrow$ negative freq. are removed. //