

Notice that I am using the properties table attached to this exam version.

(15 pts) 1. Compute the Fourier transform of the DT signal

$$x[n] = n^2 u[n-2] - n^2 u[n+2]$$

(Express your answer as a linear combination of sine and/or cosine functions.)

$$\begin{aligned} \mathcal{F}\{x[n]\} &= \sum_{n=-\infty}^{\infty} (n^2 u[n-2] - n^2 u[n+2]) e^{-j\omega n} \\ &= \sum_{n=-2}^{\infty} (-1) n^2 e^{-j\omega n} \end{aligned}$$

13

$$\begin{aligned} &= 4e^{+j\omega} - 1e^{+j\omega} + 0 - 1e^{-j\omega} - 4e^{-j\omega} \\ &= 4(e^{j\omega} + e^{-j\omega}) - (e^{j\omega} + e^{-j\omega}) \\ &= 4(\cos\omega + j\sin\omega) - (\cos\omega + j\sin\omega) \\ &= 3(\cos\omega + j\sin\omega) \end{aligned}$$

$$\boxed{X(\omega) = -8\cos(2\omega) + 2\cos\omega}$$



(15 pts) 2. Show that the Fourier transform of the CT signal $x(t) = \cos(\omega_0 t)$ is $X(\omega) = \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)$.

$$\begin{aligned}
 \mathcal{F}\{x(t)\} &= \mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\delta(\omega + \omega_0) + \delta(\omega - \omega_0)) e^{j\omega t} d\omega \\
 &= \frac{1}{2} (e^{-j\omega_0 t} + e^{j\omega_0 t}) \\
 &= \frac{1}{2} (\cos(\omega_0 t) - j\sin(\omega_0 t) + \cos(\omega_0 t) + j\sin(\omega_0 t))
 \end{aligned}$$

could
go
directly

$$= \frac{1}{2} (2 \cos(\omega_0 t))$$

$$\mathcal{F}^{-1}\{X(\omega)\} = \cos(\omega_0 t)$$

$$\mathcal{F}\{x(t)\} = \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)$$

Q.E.D.

15

(15 pts) 3. Given is a DT signal $x[n] = \frac{1}{g[n]}$ where $g[n]$ is a pure imaginary signal and an odd function of n .

a) Bob claims that the Fourier transform of $x[n]$ is $X(\omega) = \frac{1}{\cos \omega}$. Explain why Bob's answer is wrong.

$X[n]$ is real and even. This means that $X(\omega) = \mathcal{F}\{x[n]\}$ must be real and even. However, $X(\omega) = \frac{j}{\cos \omega}$ is pure imaginary. Therefore, $\frac{1}{\cos \omega}$ is not correct.

b) Alice says that the Fourier transform of $x[n]$ is $X(\omega) = \frac{1}{\sin \omega}$. Could Alice be right? Explain.

As said above, $X(\omega)$ needs to be even. However, $X(\omega) = \frac{1}{\sin \omega}$ is odd and cannot be correct.

c) Devin says that the Fourier transform of $x[n]$ is $X(\omega) = \frac{1}{\omega^2}$. Could Devin be right? Explain.

$X(\omega) = \frac{1}{\omega^2}$ is not periodic. Since $X(\omega)$ is the transform of a D.T. signal, it has to be periodic. Thus, $X(\omega) = \frac{1}{\omega^2}$ is not correct.

4. A discrete-time LTI system has frequency response

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}}$$

(15 pts) a) Derive a difference equation relating the input and the output of this system. (Use the properties of the Fourier transform listed in the table to justify your answer.)

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}}$$

$$Y(\omega) \left(1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega} \right) = 2X(\omega)$$

$$\mathcal{F}^{-1}(Y(\omega) - \frac{3}{4}e^{-j\omega}Y(\omega) + \frac{1}{8}e^{-2j\omega}Y(\omega)) = \mathcal{F}^{-1}(2X(\omega))$$

By (2.4):

$$\mathcal{F}^{-1}(Y(\omega)) - \mathcal{F}^{-1}\left(\frac{3}{4}e^{-j\omega}Y(\omega)\right) + \mathcal{F}^{-1}\left(\frac{1}{8}e^{-2j\omega}Y(\omega)\right) = 2x[n]$$

By (2.5):

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$$

15

(10 pts) b) What is the Fourier transform of the output when the input is $x[n] = (\frac{1}{4})^n u[n]$?

$$X(\omega) = \mathcal{F}\{x[n]\} = \mathcal{F}\left\{\left(\frac{1}{4}\right)^n u[n]\right\} \rightarrow \left|\frac{1}{4}\right| < 1, \text{ so by (40)}$$

$$X(\omega) = \frac{1}{1 - \frac{1}{4}e^{-j\omega}}$$

$$Y(\omega) = H(\omega)X(\omega)$$

$$Y(\omega) = \frac{2}{\left(1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}\right)\left(1 - \frac{1}{4}e^{-j\omega}\right)}$$

con: bc

10

(15 pts) b) Find the unit impulse response of this system.

$$H(z) = \frac{z}{(1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega})}$$

15

Factor!

$$q = e^{-j\omega}$$

$$(1 - \frac{3}{4}q + \frac{1}{8}q^2)$$

$$\frac{1}{8}(8 - 6q + q^2)$$

$$\frac{1}{8}(2-q)(4-q)$$

goes on scratch

$$H(z) = \frac{16}{(z-q)(4-q)}$$

Partial Fraction!

$$\frac{16}{(z-q)(4-q)} = \frac{A}{z-q} + \frac{B}{4-q}$$

$$16 = (4-q)A + (z-q)B$$

$$= 4A - Aq + zB - Bq$$

$$16 = (4A + zB) - (A+B)q$$

$$16 = 4A + zB$$

$$A+B=0 \rightarrow B=-A$$

$$\begin{aligned} 16 &= 4A - 2A \\ 16 &= 2A \\ A &= 8 \\ B &= -8 \end{aligned}$$

goes on scratch

$$H(z) = \frac{8}{z-e^{-j\omega}} + \frac{-8}{4-e^{-j\omega}}$$

$$= \frac{4}{1-\frac{1}{2}e^{-j\omega}} + \frac{-2}{1-\frac{1}{4}e^{-j\omega}}$$

$$h[n] = \mathcal{Z}^{-1}\{H(z)\} = 4 \cdot \left(\frac{1}{2}\right)^n u[n] - 2 \left(\frac{1}{4}\right)^n u[n]$$

(20 pts) 5. Use the definition of the Fourier transform (not the properties listed in the table) to prove the following Fourier transform property. OK.

$x(at+b) \xrightarrow{F} \frac{e^{j\omega b}}{-a} X\left(\frac{\omega}{a}\right)$ for any a, b real numbers with $a < 0$.

15

~~$$\mathcal{F}(x(at+b)) = \int_{-\infty}^{\infty} x(at+b) e^{-j\omega(at+b)} dt$$~~

~~$$= e^{-j\omega b} \int_{-\infty}^{\infty} x(at+b) e^{-j\omega at} dt$$~~

$$t_2 = at + b$$

$$dt_2 = a dt$$

$$\frac{dt_2}{a} = dt$$

If $t = \infty$,
 $t_2 = -\infty$, since
 $a < 0$.
 Similarly $t = -\infty \rightarrow t_2 = \infty$

$$t = \frac{t_2 - b}{a}$$

OK

$$\mathcal{F}(x(at+b)) = \int_{-\infty}^{\infty} x(at+b) e^{-j\omega t} dt$$

$$= \int_{\infty}^{-\infty} x(t_2) e^{-j\omega \frac{t_2 - b}{a}} \frac{dt_2}{a}$$

$$= \frac{e^{j\omega b}}{-a} \int_{-\infty}^{\infty} x(t_2) e^{-j\frac{\omega}{a} t_2} dt_2$$

$$\mathcal{F}(x(at+b)) = \frac{e^{j\omega b}}{-a} X\left(\frac{\omega}{a}\right)$$

Q.E.D.