ECE 544 Fall 2013
Problem Set 8
Due November 8, 2013

1. Read Chapter 7 of M. B. Pursley, *Introduction to Digital Communications* (MBP).
2. MBP Problems 6.27, 6.32, [redacted]

Defer to Prob. Set 9

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6.27 Consider seven signals constructed from the following sequences:

1110010
1100101
1001011
0010111
0101110
1011100
0111001

These sequences are generated by the shift register of Figure 6-40. The continuous-time signals are obtained from the sequences according to (6.65) of Section 6.6.1. Refer to these seven signals as $s_0, s_1, \ldots, s_6$, where s_0 is the signal corresponding to the first sequence in the list, s_1 to the second sequence, etc. Each signal has duration T, and each consists of a sequence of chips of duration T_c. The signals form a 7-ary signal set for communication over an AWGN channel with spectral density $N_0/2$. The receiver consists of a filter with impulse response $h(t)$, a sampler which samples the filter output at seven different times, and a decision device that bases decisions on the largest sample.

Construct the impulse response for a continuous-time filter that is matched to the sequence 1110010111001 . First obtain the continuous-time signal $s(t)$ from the given sequence as described in Section 6.6.1, and then let the impulse response of the filter be $h(t) = s(T_0 - t)$, $-\infty < t < +\infty$, where T_0 is an arbitrary time reference. Each of the chips for this filter has duration T_c, so $h(t)$ has duration $13T_c = 13T/7$. The value of T_0 is not critical: $T_0 = 0$ or $T_0 = 13T_c$ are acceptable choices. ($T_0 = 13T_c$ provides a causal filter.)

(a) Find the output of the filter for each of the seven signals $s_0, s_1, \ldots, s_6$.

(b) Find the optimum sampling time for each of the seven signals.

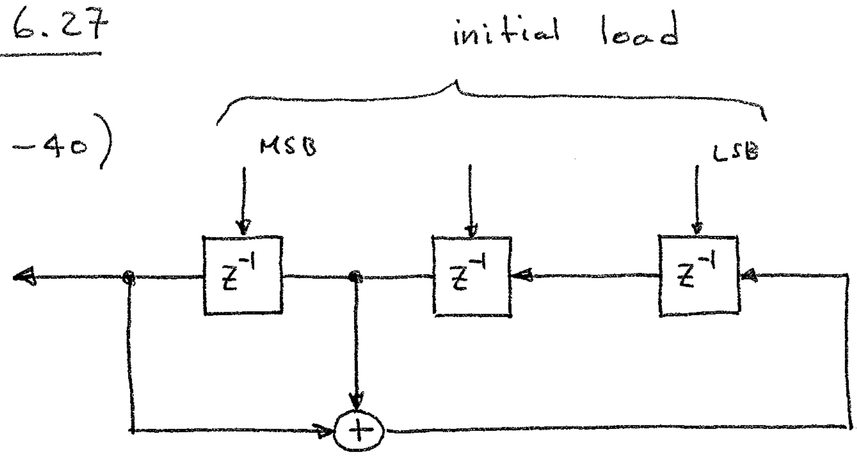
(c) How might you use the features demonstrated in part **(b)** to make symbol decisions?

(d) Consider binary signaling using s_0 and s_1 only. The receiver samples the filter output at the optimum times for signals s_0 and s_1 (from part **(b)**). Find $P_{e,i}$ for $i = 0, 1$, and compare with the error probabilities for the optimum receiver.

(e) Consider binary signaling using s_0 and s_2 only. Find $P_{e,0}$ and $P_{e,2}$ and compare your answers with the error probabilities for the optimum receiver.

MBP 6.27

(Fig. 6-40)



Mapping to signals $s_0\ s_1 \ldots s_6$

$n=0$ $n=6$

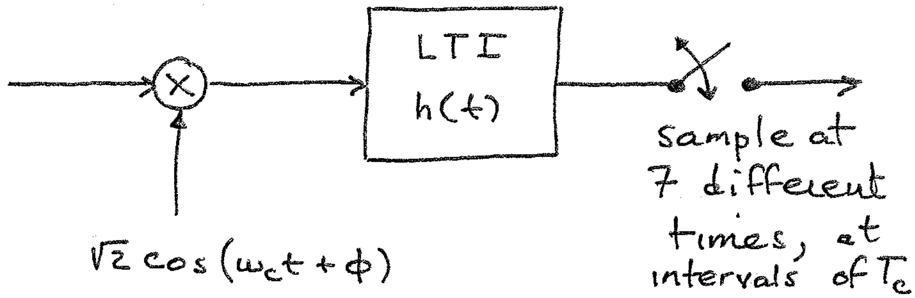
	Signal	Bit Stream $b_{i,n}$	Symbol Stream $\alpha_{i,n}$
$i=0$	s_0	1 1 1 0 0 1 0	-1 -1 -1 1 1 -1 1
	s_1	1 1 0 0 1 0 1	-1 -1 1 1 -1 1 -1
	s_2	1 0 0 1 0 1 1	-1 1 1 -1 1 -1 -1
	s_3	0 0 1 0 1 1 1	1 1 -1 1 -1 -1 -1
	s_4	0 1 0 1 1 1 0	1 -1 1 -1 -1 -1 1
	s_5	1 0 1 1 1 0 0	-1 1 -1 -1 -1 1 1
$i=6$	s_6	0 1 1 1 0 0 1	+1 -1 -1 -1 1 1 -1

$$a_i(t) = \sum_{n=0}^{6} \alpha_{i,n}\, p_{T_c}(t - nT_c) \qquad i = 0, 1, 2, \ldots 6$$

$$s_i(t) = \sqrt{2}\, A\, a_i(t) \cos(\omega_c t + \phi).$$

In this problem we don't really need to worry about the carrier $\cos(\omega_c t + \phi)$ as the problem really concerns properties of the baseband waveforms.

Hence consider a receiver of the following form



Problem says that $h(t)$ should be matched to a signal $s(t)$ obtained from

1 1 1 0 0 1 0 1 1 1 0 0 1 ← bit stream

-1 -1 -1 1 1 -1 1 -1 -1 -1 1 1 -1 ← symbol stream (α_n)

$$s(t) = \sum_{n=0}^{12} \alpha_n \, p_{T_c}(t - nT_c)$$

$h(t) = s(T_0 - t)$ where choice of T_0 is arbitrary

... $T_0 = 0$ is a purely anticausal $h(t)$; $T_0 = 13T_c$ gives a causal $h(t)$.

Problem also says that signals $s_0 \; s_1 \ldots s_6$ have duration T, which implies that

$$T = 7T_c$$

(a) Find filter outputs for every signal s_i

First look at mixer output

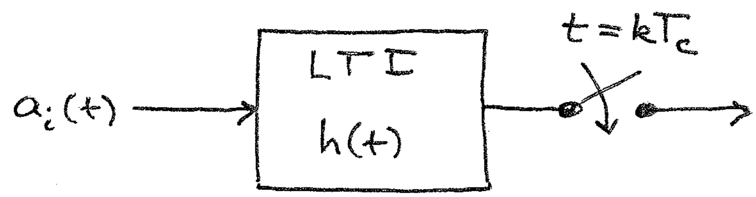
$$s_i(t)\sqrt{2}\cos(\omega_c t+\phi) = 2A\,a_i(t)\cos^2(\omega_c t+\phi)$$

$$= A\,a_i(t) + A\,a_i(t)\cos(2\omega_c t+2\phi)$$

$$= \text{input to LTI filter.}$$

Since the filter is LTI can consider inputs $A a_i(t)$ and $A a_i(t)$ ~~+~~ $A a_i(t)\cos(2\omega_c t+2\phi)$ separately. Also, <u>claim</u> that only $A a_i(t)$ will influence output. ~~We will come back to this point later.~~

Consider:



$h(t) = s(T_o - t)$ and we will take $T_o = 0$ for now because that makes it slightly simpler to line things up.

Define

$$\hat{s}_i(t) = a_i * h(t)$$

$$= \int a_i(\lambda)\,h(t-\lambda)\,d\lambda = \int a_i(\lambda)\,s(\lambda - t)\,d\lambda$$

Then sample @ $t = kT_c$

$$\hat{s}_i(kT_c) = \int a_i(\lambda)\,s(\lambda - kT_c)\,d\lambda$$

Plugging in for definitions of $a_i(\lambda)$ and $s(\lambda - kT_c)$ have

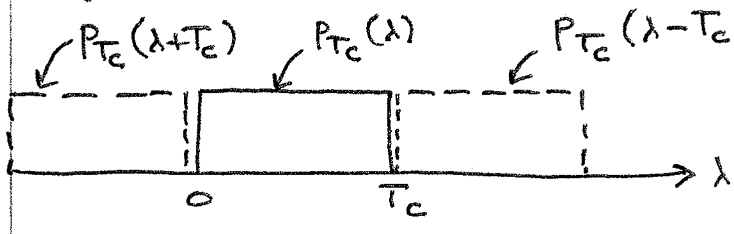
$$(*)\quad \hat{S}_i(kT_c) = \sum_{n=0}^{6} \sum_{m=0}^{12} \alpha_{i,n}\, \alpha_m \underbrace{\int p_{T_c}(\lambda - nT_c)\, p_{T_c}(\lambda - kT_c - mT_c)\, d\lambda}$$

$$\lambda' = \lambda - nT_c$$

$$\int p_{T_c}(\lambda')\, p_{T_c}(\lambda' + nT_c - kT_c - mT_c)\, d\lambda'$$

Thus consider integrals of the form

$$\int p_{T_c}(\lambda)\, p_{T_c}(\lambda - \ell T_c)\, d\lambda = \begin{cases} T_c & \ell = 0 \\ 0 & \ell \neq 0 \end{cases}$$



$\therefore$ no overlap unless $\ell = 0$

Back to $\hat{S}_i(kT_c)$ calculation.

In the sum over m in Eq. $(*)$ only term $m = n - k$ is nonzero. Hence

$$\hat{S}_i(kT_c) = T_c \sum_{n=0}^{6} \alpha_{i,n}\, \alpha_{n-k}$$

If we extend the sequences $\alpha_{i,n}$ and α_n by prepending and appending with zero we can rewrite as

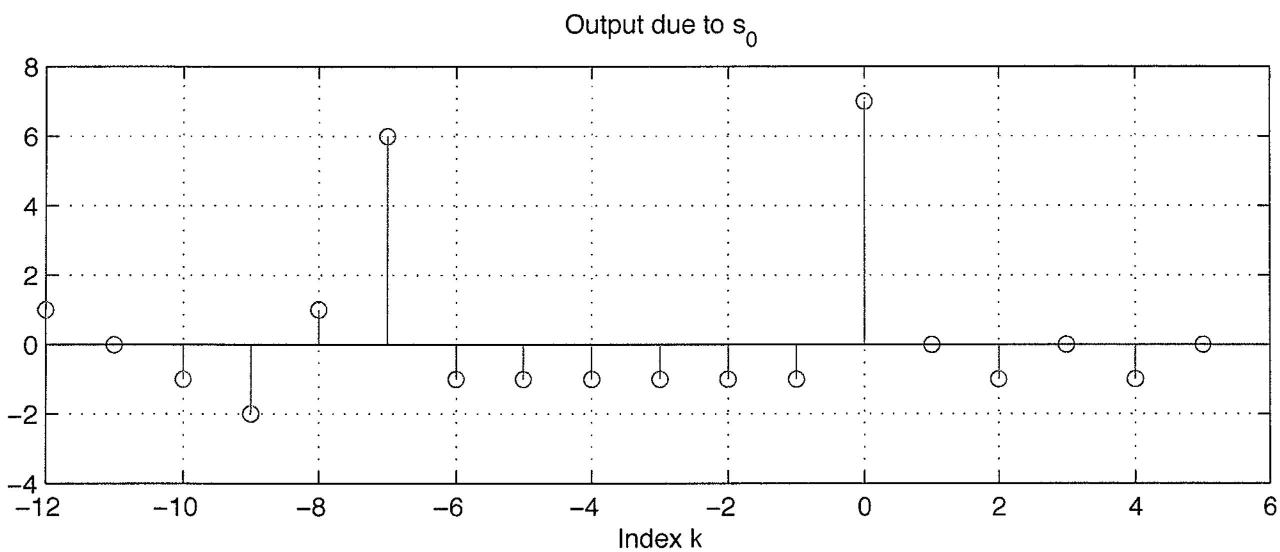
$$\hat{S}_i(kT_c) = T_c \sum_{n} \alpha_{i,n}\, \alpha_{n-k} = T_c \sum_{m} \alpha_{i,m+k}\, \alpha_m$$

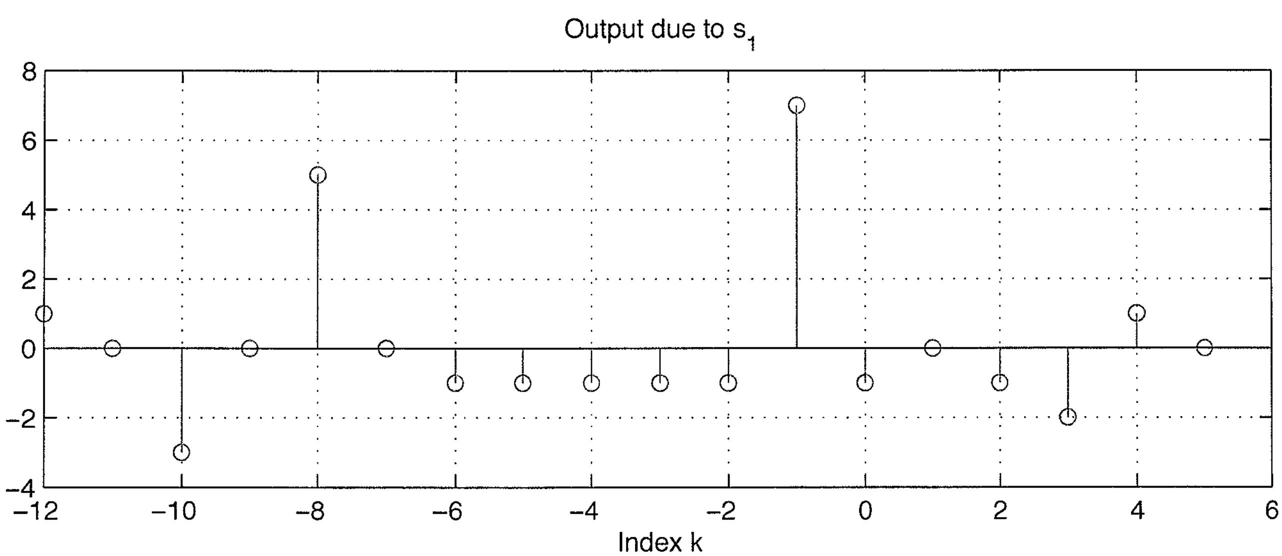
which is the cross-correlation of the two sequences.

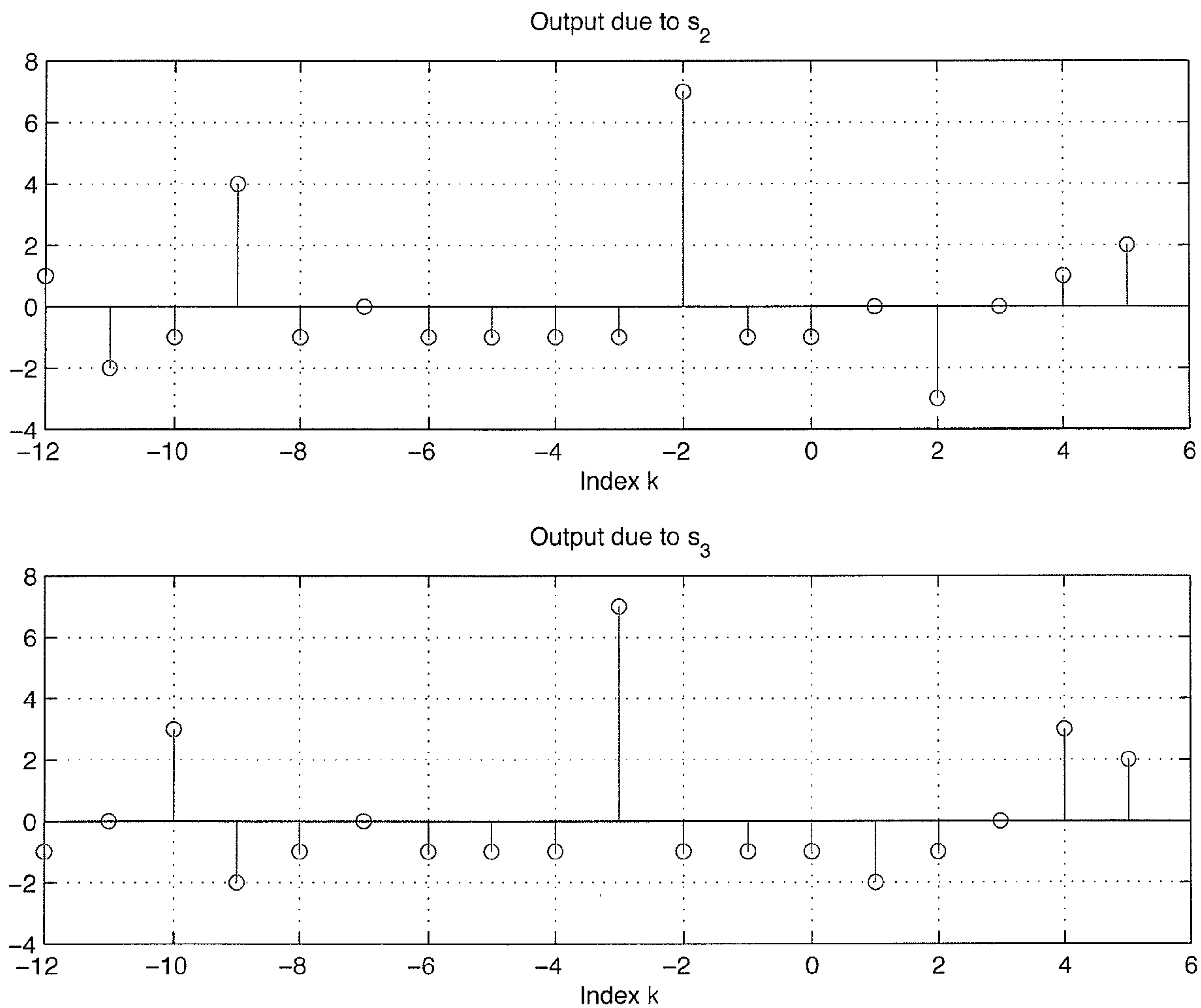
Implementing these cross-correllations (one for each signal $i = 0, 1, 2 \dots 6$) is possible to do manually, but much easier via Matlab. This was done and results are plotted. See following pages.

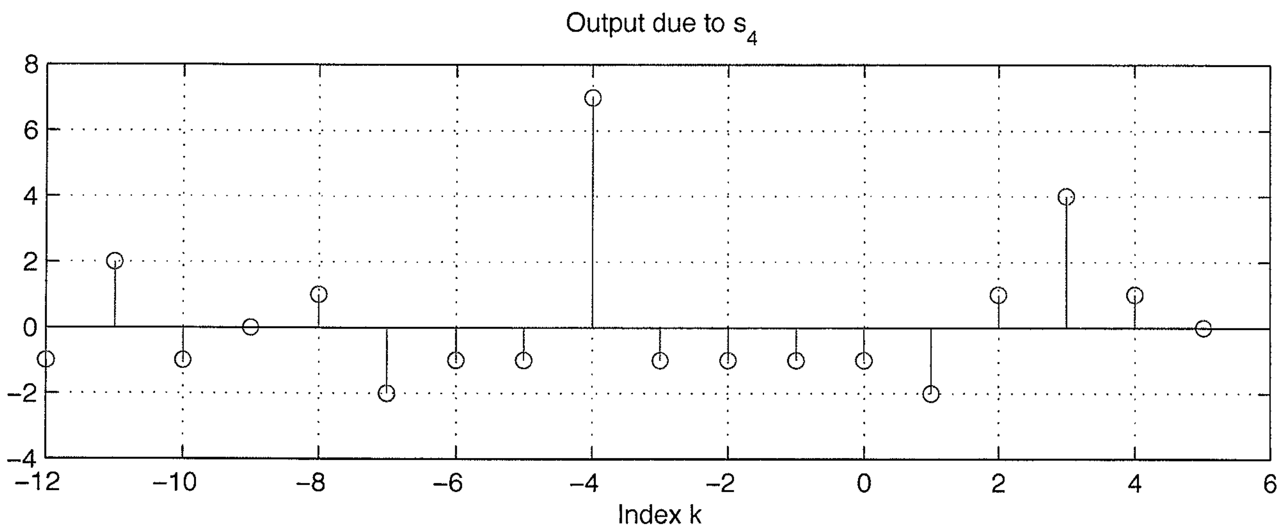
If instead of the non-causal MF we had chosen $T_0 = 13T_c$ this would only shift the outputs relative to index k. In this case the plots would start with first point at $k=1$ (rather than $k=-12$ as plotted).

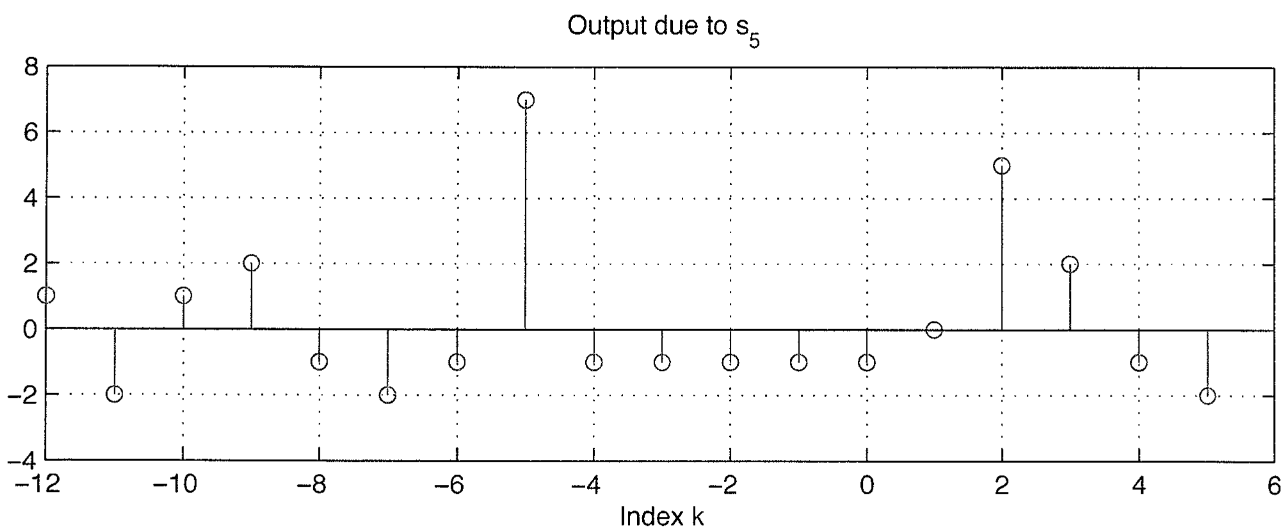
The actual outputs of the LTI filter would be the integers shown in the plots multiplied by AT_c.

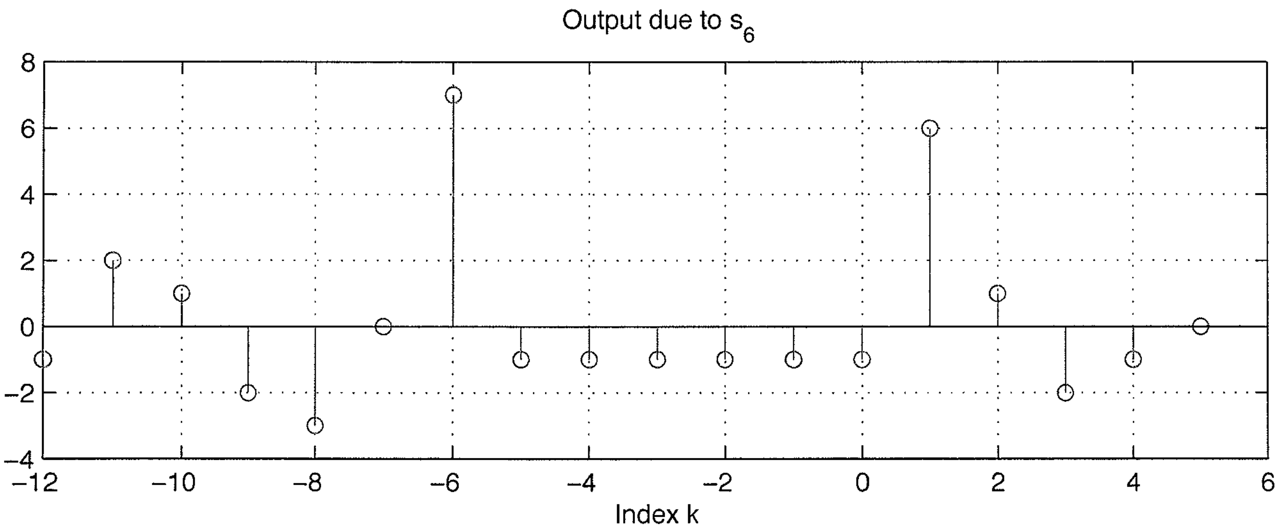

Output due to s_0
8
6
4
2
0
−2
−4
−12
−10
−8
−6
−4
−2
0
2
4
6
Index k


Output due to s_1
8
6
4
2
0
−2
−4
−12
−10
−8
−6
−4
−2
0
2
4
6
Index k


Output due to s_2
8
6
4
2
0
−2
−4
−12
−10
−8
−6
−4
−2
0
2
4
6
Index k
Output due to s_3
8
6
4
2
0
−2
−4
−12
−10
−8
−6
−4
−2
0
2
4
6
Index k


Output due to s_4
8
6
4
2
0
−2
−4
−12
−10
−8
−6
−4
−2
0
2
4
6
Index k


Output due to s_5
8
6
4
2
0
−2
−4
−12
−10
−8
−6
−4
−2
0
2
4
6
Index k


Output due to s_6
8
6
4
2
0
−2
−4
−12
−10
−8
−6
−4
−2
0
2
4
6
Index k

(b) Opt. Sampling Times for each of seven signals
$s_0 \quad s_1 \quad \ldots \quad s_6$

These are the values that maximize the noiseless filter outputs:

Signal	k_{opt} for $T_0 = 0$	k_{opt} for $T_0 = 13T_c$
s_0	0	13
s_1	-1	12
s_2	-2	11
s_3	-3	10
s_4	-4	9
s_5	-5	8
s_6	-6	7

Notice that the continuous time output consists of straight line segments joining the samples prev. plotted.

(c) The receiver would sample at the seven times indicated. Then it would decide which of the seven samples was largest and choose as its estimate of the transmitted signal the one corresponding to the index of the largest sample.

(d) Let $T_0 = 0$. The opt. sampling times for s_0 and s_1 are

$$s_0 \longrightarrow k = 0$$

$$s_1 \longrightarrow k = -1$$

So we build the binary receiver on these two samples only (since the receiver front end is already picked for us).

If s_0 is sent:

$$A\hat{s}_0(0 \cdot T_c) = 7AT_c$$
$$A\hat{s}_0(-1 \cdot T_c) = -AT_c$$

If s_1 is sent:

$$A\hat{s}_1(0 \cdot T_c) = -AT_c$$
$$A\hat{s}_1(-1 \cdot T_c) = 7AT_c$$

Above are the signal components at the output of the sampler at the two times indicated. There will also be noise samples.

The noise input is AWGN $X(t)$ with $N_0/2$. The output noise is

$$\hat{X}(t) = \int X(\lambda) h(t-\lambda)\, d\lambda$$

Clearly $\hat{X}$ is Gaussian and of mean zero. Its autocorrelation function is

$$R_{\hat{X}\hat{X}}(\tau) = \frac{N_0}{2}\int h(\lambda) h(\lambda-\tau)\, d\lambda$$
$$= \frac{N_0}{2}\int s(\lambda) s(\lambda-\tau)\, d\lambda$$

We really only need the samples of this autocorr.

$$R_{\hat{X}\hat{X}}(kT_c) = \frac{N_0}{2}\int s(\lambda) s(\lambda - kT_c)\, d\lambda$$

$$R_{\hat{X}\hat{X}}(kT_c) = \sum_{n=0}^{12} \sum_{m=0}^{12} \alpha_n \alpha_m \underbrace{\int p_{T_c}(\lambda - nT_c)\, p_{T_c}(\lambda - kT_c - mT_c)\, d\lambda}_{= \begin{cases} T_c & m = n-k \\ 0 & \text{else} \end{cases}}$$

$$= \frac{N_0}{2} T_c \sum_{n=0}^{12} \alpha_n \alpha_{n-k}$$

We will need the samples for $k = 0, \pm 1$

$$R_{\hat{X}\hat{X}}(0) = \frac{N_0}{2} T_c \cdot 13$$

$$R_{\hat{X}\hat{X}}(1) = \frac{N_0}{2} T_c \left[0 \right] = 0 = R_{\hat{X}\hat{X}}(-1)$$

$$\left(\begin{array}{cccccccccccccc} -1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & \\ & -1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 \\ \hline 0 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 0 \end{array} \right)$$

sum to zero

For $P_{e,0}$

The receiver compares

$$\hat{Y}(0 \cdot T_c) = A\hat{S}_0(0 \cdot T_c) + \hat{X}(0 \cdot T_c)$$

to

$$\hat{Y}(-1 \cdot T_c) = A\hat{S}_0(-1 \cdot T_c) + \hat{X}(-1 \cdot T_c)$$

and selects S_1 if latter is largest.

$$\therefore P_{e,0} = P\left(A\hat{s}_0(-T_c) + \hat{X}(-T_c) > A\hat{s}_0(0) + \hat{X}(0)\right)$$

$$= P\left(\hat{X}(-T_c) - \hat{X}(0) > A\hat{s}_0(0) - A\hat{s}_0(-T_c)\right)$$

$$= P\Big(\underbrace{\hat{X}(-T_c) - \hat{X}(0)}_{} > 8AT_c\Big)$$

rv is Gaussian of mean zero and var

$$E\left\{\left(\hat{X}(-T_c) - \hat{X}(0)\right)^2\right\}$$

$$= 2R_{\hat{X}\hat{X}}(0) - 2R_{\hat{X}\hat{X}}(T_c)$$

$$= 2\frac{N_0}{2} 13T_c - 0 = N_0 13 T_c$$

$$P_{e,0} = Q\left(\frac{8AT_c}{\sqrt{N_0 13 T_c}}\right) = Q\left(8A\sqrt{\frac{T_c}{13N_0}}\right)$$

<u>For $P_{e,1}$</u>

$$\hat{Y}(0\cdot T_c) = A\hat{s}_1(0\cdot T_c) + \hat{X}(0\cdot T_c)$$

$$\hat{Y}(-T_c) = A\hat{s}_1(-T_c) + \hat{X}(-T_c)$$

$$P_{e,1} = P\left(A\hat{s}_1(0) + \hat{X}(0) > A\hat{s}_1(-T_c) + \hat{X}(-T_c)\right)$$

$$= P\left(\hat{X}(-T_c) - \hat{X}(0) < A\hat{s}_1(0) - A\hat{s}_1(-T_c)\right)$$

$$= P\left(\hat{X}(-T_c) - \hat{X}(0) < -8AT_c\right)$$

$$= Q\left(8A\sqrt{\frac{T_c}{13N_0}}\right) = P_{e,0}.$$

<u>The optimum binary receiver for these two signals</u>

The opt. receiver would be designed for the two signals s_0 and s_1. They each have energy

$$7A^2T_c$$

and their cross-correlation is $-A^2T_c$. Therefore, their correlation coeff is $r = -1/7$. We can use

$$Q\left(\sqrt{\frac{\mathcal{E}(1-r)}{N_0}}\right)$$

for the error prob.

$$P_{e,0}^{opt} = P_{e,1}^{opt} = Q\left(\sqrt{\frac{7A^2T_c(8/7)}{N_0}}\right)$$

$$= Q\left(A\sqrt{\frac{8T_c}{N_0}}\right) = Q\left(8A\sqrt{\frac{T_c}{8N_0}}\right)$$

Thus opt. is better by

$$10\log_{10}(13/8) = 2.1 \text{ dB}$$

(e) Same idea. However the samples of interest are now $k=0$ and $k=-2$ and it requires to compute $R_{\hat{x}\hat{x}}(2T_c) = -3N_0T_c$ which incr. the noise variance.

$$P_{e,0} = P_{e,1} = Q\left(2A\sqrt{T_c/N_0}\right)$$

and the loss from opt. is now 3dB.

6.32 Consider the two signal sets illustrated in Figure 6-54. Assume that a maximum-likelihood receiver is used for each signal set, and the channel is an additive white Gaussian noise channel with two-sided spectral density $N_0/2$. The distance between pairs of signals is d for each set. The signals in Set 1 are equal-energy, orthogonal, three-dimensional signals. The signals in Set 2 are equal-energy, two-dimensional signals. Let $\mathcal{E}_1$ denote the energy for each signal in Set 1, and let $\mathcal{E}_2$ denote the energy for each signal in Set 2.

Fact: Since the distance between signals is the same for the two sets, the symbol error probability for maximum-likelihood reception is also the same for the two sets.

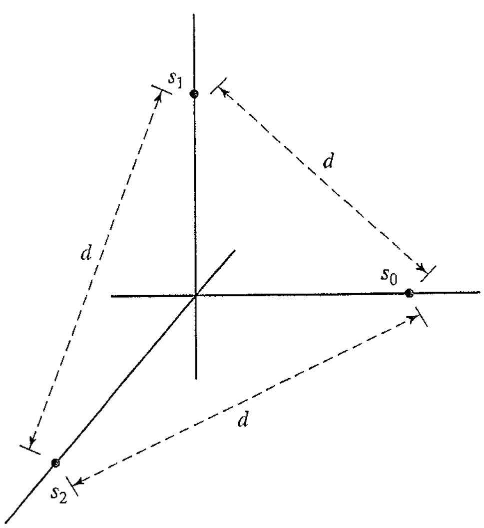
(a) Give an expression for d in terms of $\mathcal{E}_1$.

(b) For Set 1, use (6.72) in Section 6.6.3 and your result in part **(a)** to obtain an expression for the probability of symbol error in terms of d.

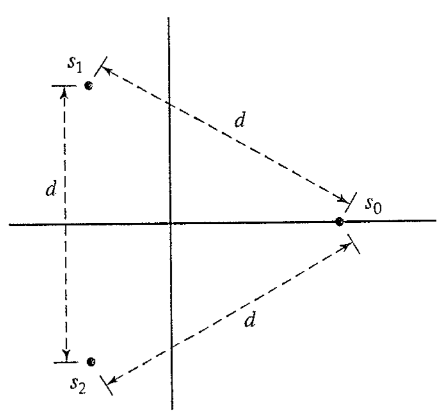
(c) Use simple trigonometry to find an expression for $\mathcal{E}_2$ in terms of d. *Hint*: d is the length of each side of the equilateral triangle whose vertices are the points in the constellation for Set 2. From this hint, it is easy to determine various angles that may be useful in the evaluation of $\mathcal{E}_2$.

(d) Use the fact in the problem statement and your results from parts **(b)** and **(c)** to obtain an expression for the probability of symbol error in terms of $\mathcal{E}_2$ for Set 2. For a given symbol error probability, which set requires less energy?

(e) Give a numerical value for the correlation coefficient r for pairs of signals in Set 2. What type of signal set is Set 2?



(a) Set 1 (orthogonal signals)



(b) Set 2

Figure 6-54: Signal sets for Problem 6.32.

MBP 6.32

(a) For Set 1 let $\underline{s_0} = (\alpha, 0, 0)$ and note that $\underline{s_1} = (0, \alpha, 0)$ and $\underline{s_2} = (0, 0, \alpha)$.

$$\|\underline{s_0}\|^2 = \alpha^2 = \mathcal{E}_1$$

From Mr. Pythagoras $\quad \alpha^2 + \alpha^2 = 2\alpha^2 = d^2$

$$\Longrightarrow \alpha^2 = d^2/2$$

$$\therefore \quad \frac{d^2}{2} = \mathcal{E}_1 \Longrightarrow d = \sqrt{2\mathcal{E}_1}$$

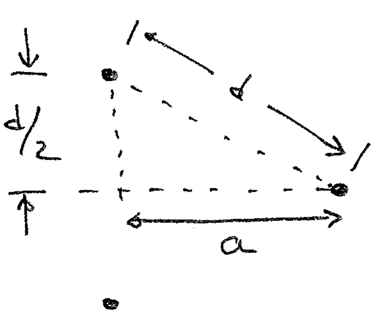
(b) Eq. (6.72) is

$$P_e = (M-1)\int_{-\infty}^{\infty} \left[\Phi(v)\right]^{M-2} \Phi\left(v - \sqrt{2\mathcal{E}/N_0}\right) e^{-v^2/2} \frac{dv}{\sqrt{2\pi}}$$

In this case $M = 3$, $\mathcal{E} = \mathcal{E}_1 = d^2/2$

$$P_e = 2\int_{-\infty}^{\infty} \Phi(v)\, \Phi\left(v - d/\sqrt{N_0}\right) e^{-v^2/2} \frac{dv}{\sqrt{2\pi}}$$

(c) For Set 2 the triangle is equilateral. All angles are 60°



$$a^2 + \frac{d^2}{4} = d^2$$

$$a^2 = \frac{3}{4} d^2$$

$$a = \frac{\sqrt{3}}{2} d$$

The points $\underline{s_0}$ $\underline{s_1}$ $\underline{s_2}$ are on a circle cent. at $(0,0)$ with radius $\sqrt{\varepsilon_2}$.

$$\underline{s_0} = (\sqrt{\varepsilon_2}, 0)$$

$$\underline{s_1} = (\beta, d/2)$$

$$\underline{s_2} = (\beta, -d/2)$$

$$\|\underline{s_1}\|^2 = \beta^2 + \frac{d^2}{4} = \varepsilon_2 \Rightarrow \beta = \sqrt{\varepsilon_2 - d^2/4}$$

Must also have

$$\beta = \sqrt{\varepsilon_2} - a = \sqrt{\varepsilon_2} - \frac{\sqrt{3}}{2} d$$

Putting these together

$$\cancel{\varepsilon_2} - \sqrt{3} d \sqrt{\varepsilon_2} + \frac{3}{4} d^2 + \frac{d^2}{4} = \cancel{\varepsilon_2}$$

$$-\sqrt{3\varepsilon_2}\, d + d^2 = 0$$

$$d(d - \sqrt{3\varepsilon_2}) = 0 \Longrightarrow d = \sqrt{3\varepsilon_2}$$

$$d^2 = 3\varepsilon_2$$

$$\varepsilon_2 = \frac{1}{3} d^2 = \frac{2}{3} \frac{d^2}{2} = \frac{2}{3} \varepsilon_1$$

(d) Since d is same for Set 1 and Set 2 and using the "Fact" and $d = \sqrt{3\mathcal{E}_2}$ we can write

$$P_e = 2\int_{-\infty}^{\infty} \Phi(v)\,\Phi\left(v - \sqrt{3\mathcal{E}_2/N_0}\right) e^{-v^2/2}\,\frac{dv}{\sqrt{2\pi}}$$

Set 2 requires only $\frac{2}{3}$ of the energy of Set 1 for the same symbol error prob.

(e)

$$\langle \underline{s}_0, \underline{s}_1 \rangle = \mathcal{E}_2 \cos(2\pi/3) = -\mathcal{E}_2/2$$

$$\therefore\ r = \langle \underline{s}_0, \underline{s}_1 \rangle / \mathcal{E}_2 = -1/2$$

This is a <u>simplex signal set.</u>