7/29 ①

- Freq. Response of DT LTI Systems described by diffence equations.
  - Digital filter : difference equation implemented in either software or hardware.
  - used to perform freq selective filtering and many other signal processing functions.

<general form of difference eq>

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

Interested in the freq response of an LTI system (= difference eq), which is the DTFT of the impulse response of the system.

$$h[n] \xleftrightarrow{DTFT} H(w)$$

- Three properties of DTFT — linearity, time-shift, convolution — allow us to find $H(\omega)$ without ever determining $h[n]$ (impulse response).
- Take DTFT of both sides of Diff. eq. using linearity and time-shift properties.

$$Y(\omega) = -\sum_{k=1}^{N} a_k e^{-jk\omega} Y(\omega) + \sum_{k=0}^{M} b_k e^{-jk\omega} X(\omega)$$

$$Y(\omega)\left(1+\sum_{k=1}^{N} a_k e^{-jk\omega}\right) = \left(\sum_{k=0}^{M} b_k e^{-jk\omega}\right) X(\omega)$$

$$Y(\omega) = \underbrace{\left\{\frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{1+\sum_{k=1}^{N} a_k e^{-jk\omega}}\right\}}_{=H(\omega)} X(\omega)$$

<General Info> LTI

$$\left(\begin{array}{l} y[n] = h[n] * x[n] \\ Y(\omega) = H(\omega)X(\omega) \end{array}\right)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{1+\sum_{k=1}^{N} a_k e^{-jk\omega}} \quad : \text{"Frequency Response"}$$

$\updownarrow$ I-DTFT

$h[n]$

$$= \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}, \quad a_0 = 1$$

$a_1 = -1$ $\quad b_0 = 1$ $\quad b_4 = -1$ $\quad b_1 = b_2 = b_3 = 0$

Example $\quad y[n] = y[n-1] + x[n] - x[n-4]$

(method 1) $\quad H(\omega) = \dfrac{1 - e^{-j4\omega}}{1 - e^{-j\omega}}$ //

(method 2) Taking DTFT of the diff eq,

$$Y(\omega) = Y(\omega) e^{-j\omega} + X(\omega) - X(\omega) e^{-j4\omega}$$

$$Y(\omega)(1 - e^{-j\omega}) = X(\omega)(1 - e^{-j4\omega})$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1 - e^{-j4\omega}}{1 - e^{-j\omega}} \;//$$

- Use "half-angle trick" to evaluate how system responds at different freq.

$$H(\omega) = \frac{(e^{j2\omega} - e^{-j2\omega})\, e^{-j2\omega}}{(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}})\, e^{-j\frac{\omega}{2}}} \cdot \frac{\frac{1}{2j}}{\frac{1}{2j}}$$

$$= \frac{\sin(2\omega)}{\sin(\frac{\omega}{2})} e^{-j\frac{3}{2}\omega}$$

"DIY" ◦ compare this with DTFT of $h[n] = u[n] - u[n-4]$

Prob 5.21 (b) $x[n] = \left(\frac{1}{2}\right)^{-n} u[-n-1] \overset{DTFT}{\longleftrightarrow} X(\omega) = ?$

$$= \left(\frac{1}{2}\right)^{-n} u[-(n+1)]$$

$$= \frac{1}{2} \underline{\left(\frac{1}{2}\right)^{-(n+1)} u[-(n+1)]}$$

using
① $a^n u[n] \overset{DTFT}{\longleftrightarrow} \frac{1}{1-ae^{-j\omega}}$
② $x[-n] \overset{DTFT}{\longleftrightarrow} X(-\omega)$
③ $x[n-n_0] \overset{DTFT}{\longleftrightarrow} e^{-j\omega n_0} X(\omega)$

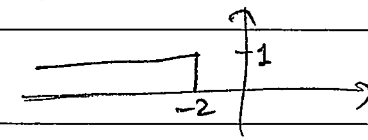
$$X(\omega) \Rightarrow \frac{1}{2} \frac{1}{1-\frac{1}{2}e^{-j\omega}} e^{-j\omega(-1)}$$

$$\Rightarrow \frac{1}{2} \frac{1}{1-\frac{1}{2}e^{-j(-\omega)}} e^{-j(-\omega)(-1)}$$

Prob 5.24 (c) $\quad x[n] = \left(\frac{1}{3}\right)^{|n|} u[-n-2] \xleftrightarrow{\text{DTFT}} X(\omega) = ?$

$$= \left(\frac{1}{3}\right)^{|n|} \underbrace{u[-(n+2)]}$$

turns on at $n = -\infty$ shuts off at $n = -1$



$x[n] = \left(\frac{1}{3}\right)^{-n} u[-(n+2)]$ since $|n| = -n$ for $n < 0$

$$= \left(\frac{1}{3}\right)^{2} \left(\frac{1}{3}\right)^{-(n+2)} u[-(n+2)]$$

From the previous problem,

$$X(\omega) = \frac{1}{9} e^{-j\omega(-2)} \frac{1}{1 - \frac{1}{3} e^{-j(-\omega)}} \quad //$$