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Problem 114: A local ring is a commutative ring with 1 that has a unique maximal ideal. Show that a ring R is local if and only if the set of non-units in R is an ideal.

Let R be a commutative ring with 1. Let $r \in R$ be a non-unit. Then, by Proposition 11, Chapter 7, $(r) \subseteq M_r$ where $M_r$ is some maximal ideal in R. Let M be an ideal in R.

M is the unique maximal ideal $\Rightarrow$M$_r$ $=$ M $\forall r \in R$ non-units $\Rightarrow$r$\in M$ $\forall r \in R$ non-units, and in fact there are no units in M (otherwise M=R and M is not maximal), so M is the set of non-units in R and an ideal.

On the other hand, if M is the set of non-units in R and an ideal, then if S is maximal ideal in R, S cannot contain a unit. Therefore, $S \subseteq M$, and by maximality S=M and M is the unique maximal ideal.