Functions of R.V.

Let X be a r.v. and $g(x)$ a function defined on the real line.
Define $Y = g(X)$. Then Y is also a r.v.

Would like to find $F_Y(y)$, $f_Y(y)$, $P_Y(y_j)$ from $F_X(x)$, $f_X(x)$, $P_X(x_i)$ and $g(x)$.

Functions of Discrete R.V.

If X is discrete, it takes on values from
$S_X = \{x_i : i=1, \dots, N\}$
Since $Y = g(X)$, we must have that Y takes on values from

$$S_Y = \{g(x_i) : i=1,\dots,N\}$$
$$= \{y_j : j = 1, \dots, M\} \qquad (M \leq N)$$

Therefore Y must be a discrete r.v.

$$P_Y(y_j) = \Pr(Y = y_j)$$
$$= \Pr(g(X) = y_j)$$
$$= \Pr(X \in \{x_i : g(x_i) = y_j\})$$

$$\Rightarrow \boxed{P_Y(y_j) = \sum_{x_i : g(x_i) = y_j} P_X(x_i)}$$

Ex Let X be a discrete r.v. with pmf

$$P_X(x_i) = \begin{cases} 1/8, & x_i = -1, -2 \\ 1/4, & x_i = 0 \\ 1/2, & x_i = 1 \end{cases}$$

Let $g(x) = |x|$ and define $Y = g(X) = |X|$

Find the pmf of Y

$S_X = \{-2, -1, 0, 1\}$

$S_Y = \{ |x_i| : x_i = -2, -1, 0, 1 \}$

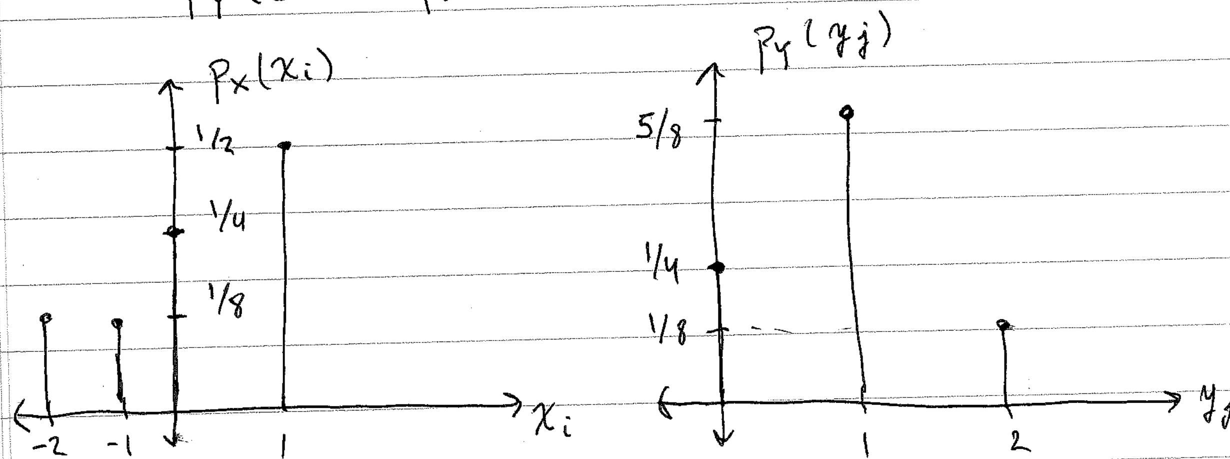
$= \{0, 1, 2\}$

$\cancel{P_Y(y_j) = P_X}$

$P_Y(0) = P_X(0) = 1/4$

$P_Y(1) = P_X(1) + P_X(-1) = 5/8$

$P_Y(2) = P_X(-2) = 1/8$



Functions of Continuous R.V.

When X is a continuous r.v., Y may be discrete, continuous, or mixed depending on $g(x)$.

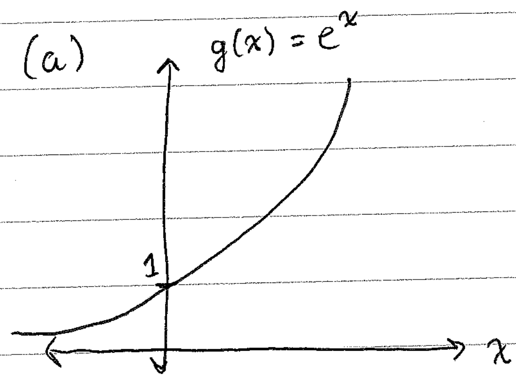
Review some properties and definitions of functions

$g(x)$ is smooth if $g'(x)$ exists and is continuous for all $x \in \mathbb{R}$

$g(x)$ is piecewise smooth if the above is true except at a finite or countable number of points.

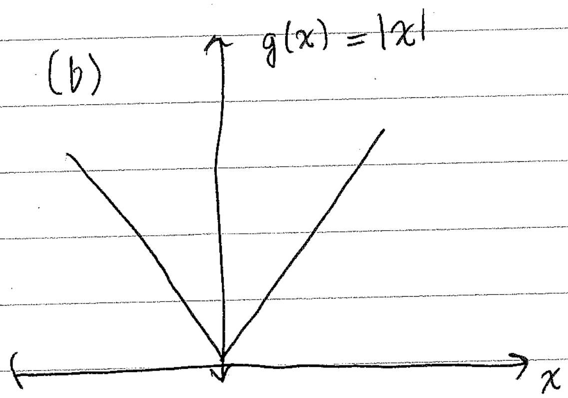
$g(x)$ is one-to-one (1:1) if $x_1 \neq x_2 \Rightarrow g(x_1) \neq g(x_2)$

Ex (a)



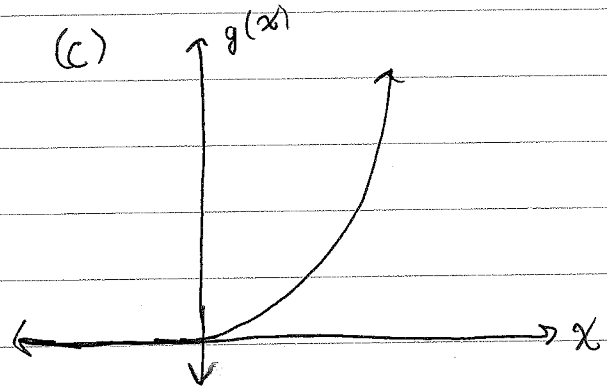
$g(x)$ is 1:1 and smooth

(b)



$g(x)$ is not 1:1 but N:1 (2:1 for $x \neq 0$)
$g(x)$ is piece wise smooth

(c)



$g(x)$ is not 1:1 and not N:1

$g(x)$ smooth

In (a) $\cancel{g(x)}$ $g(X)$ will yield a continuous r.v. since $Pr(Y=y) = Pr(X=x) = 0$

In (b) $\cancel{g(x)}$ $g(X)$ will yield a continuous r.v. since $Pr(Y=y) = Pr(X=x_1) + Pr(X=x_2) = 0$

In (c) $g(X)$ will yield a ~~mixed~~ continuous (or discrete) r.v. since $Pr(Y=0) = Pr(X \le 0) = F_X(0)$ which may not be 0.

Will always assume that $g(x)$ is at least piecewise smooth.

Will give two methods for the case that X is a continuous r.v. : Distribution Method and the Density Method.

Distribution Method

$$F_Y(y) = \Pr(Y \le y)$$

$$= \Pr(g(X) \le y)$$

$$= \Pr(X \in \{\cancel{g(x)}\; x : g(x) \le y\})$$

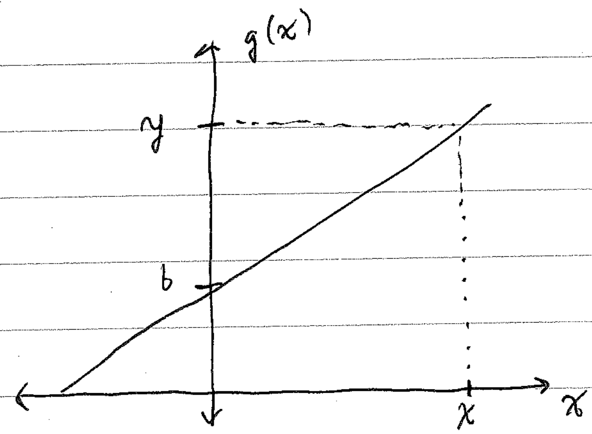
$$\boxed{\begin{aligned} F_Y(y) &= \int_{x: g(x) \le y} f_X(x)\, dx \\ f_Y(y) &= \frac{d}{dy} F_Y(y) \end{aligned}}$$

For each y, need to find $\{x : g(x) \le y\}$

Ex: $g(x) = ax + b$, $(a > 0)$

X is a continuous r.v. and $Y = g(X)$.

Find $f_Y(y)$ in terms of $f_X(x)$, a, and b

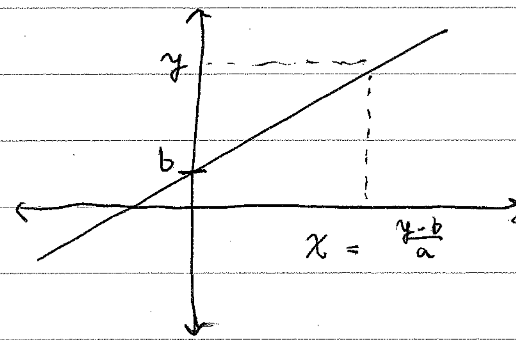


$$= \frac{y-b}{a}$$

<u>Ex</u> $g(X) = ax + b$, $(a > 0)$

Let X be a continuous r.v. and define $Y = g(X)$. Find $f_Y(y)$ in terms of $f_X(x)$

<u>Distribution Method:</u>



~~F_Y~~

$$F_Y(y) = \int_{g(x) \le y} f_X(x)\,dx$$

$$= \int_{-\infty}^{\frac{y-b}{a}} f_X(x)\,dx$$

$$\left(\frac{d}{dy} \int_{a(y)}^{b(y)} f(x)\,dx = f(b(y)) \cdot \frac{d}{dy} b(y) - f(a(y)) \cdot \frac{d}{dy} a(y) \right.$$

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$= f_X\left(\frac{y-b}{a}\right) \cdot \frac{d}{dy} \frac{y-b}{a}$$

$$= f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a}, \quad -\infty < y < \infty$$

Assume $f_X(x) = e^{-x} u(x)$

$$F_Y(y) = \int_{-\infty}^{\frac{y-b}{a}} \cancel{f_X(x)}\, e^{-x} u(x)\,dx$$

$$= \int_{0}^{\frac{y-b}{a}} e^{-x}\,dx, \quad \frac{y-b}{a} \ge 0 \iff (y \ge b)$$

$$= 1 - e^{-\left(\frac{y-b}{a}\right)}, \quad \cancel{\tfrac{y-b}{a}} \; y \ge b$$

$$F_Y(y) = 0, \quad y < b$$

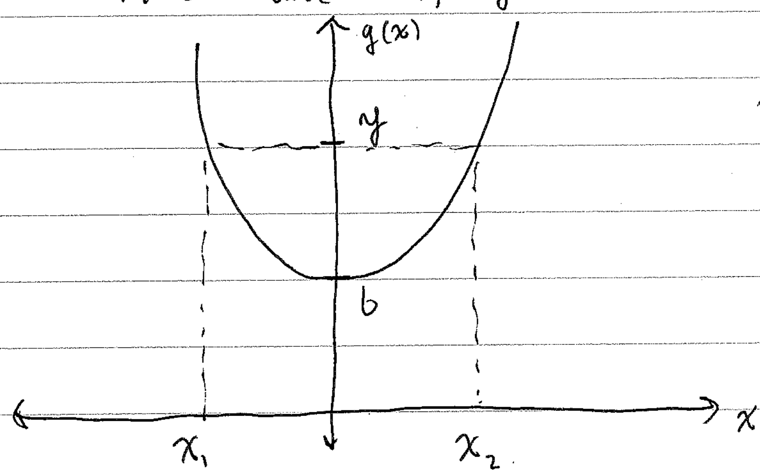
$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$= \begin{cases} \exp\left(-\frac{y-b}{a}\right) & , \; y \geq b \\ 0 & , \; \text{else} \end{cases}$$

Ex: $g(x) = ax^2 + b$, $a > 0$

X is a continuous r.v. , $Y = g(X)$

Find ~~the~~ $f_Y(y)$



$$x_1 = -\sqrt{\frac{y-b}{a}}$$

$$x_2 = \sqrt{\frac{y-b}{a}}$$

Distribution Method

$$F_Y(y) = \int_{x:\, g(x) \leq y} f_X(x)\, dx$$

$$= \int_{-\sqrt{\frac{y-b}{a}}}^{\sqrt{\frac{y-b}{a}}} f_X(x)\, dx \quad , \; y > b$$

$$= 0 \quad , \; y \leq b$$

$$F_Y(y) = \int_{x:\, g(x) \le y} f_X(x)\, dx$$

$$= \int_{-\infty}^{\frac{y-b}{a}} f_X(x)\, dx$$

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

Leibniz Rule: $\displaystyle \frac{d}{dy} \int_{a(y)}^{b(y)} f(x)\, dx = f(b(y)) \cdot \frac{d}{dy} b(y)$

$$- f(a(y)) \cdot \frac{d}{dy} a(y)$$

$$f_Y(y) = f_X\left(\frac{y-b}{a}\right) \cdot \frac{d}{dy} \frac{y-b}{a}$$

$$= f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a} \quad , \quad y \in \mathbb{R}$$

Assume $f_X(x) = e^{-x} u(x)$

$$= \begin{cases} e^{-x} & , \quad x \ge 0 \\ 0 & , \quad \text{else} \end{cases}$$

$$F_Y(y) = \int_{-\infty}^{\frac{y-b}{a}} e^{-x} u(x)\, dx$$

$$= \int_0^{\frac{y-b}{a}} e^{-x}\, dx \quad , \quad \frac{y-b}{a} \ge 0 \iff y \ge b$$

$$= 1 - \exp\left(-\frac{y-b}{a}\right) \quad , \quad y \ge b$$

$$F_Y(y) = 0 \quad , \quad y < b$$

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$= f_X\left(\sqrt{\frac{y-b}{a}}\right) \frac{d}{dy} \sqrt{\frac{y-b}{a}}$$

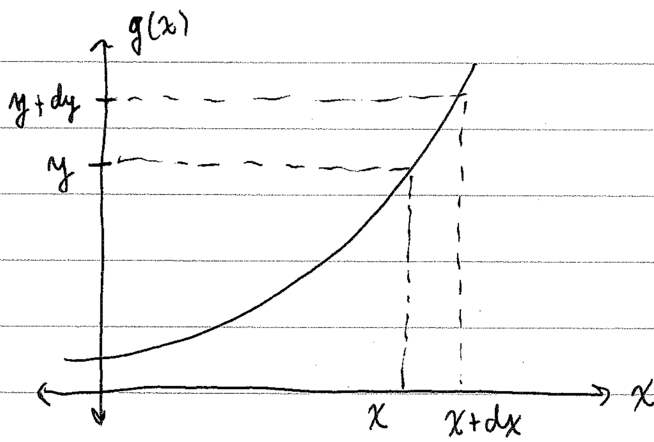
$$- f_X\left(-\sqrt{\frac{y-b}{a}}\right) \frac{d}{dy} - \sqrt{\frac{y-b}{a}} \quad , \quad y > b$$

$$= \left(f_X\left(\sqrt{\frac{y-b}{a}}\right) + f_X\left(-\sqrt{\frac{y-b}{a}}\right)\right) \cdot \frac{1}{2\sqrt{a(y-b)}} \quad , \quad y > b$$

$$= 0, \cancel{et} \qquad , \quad y \le b$$

Density Method

Consider a 1:1 function $g(x)$ with $g'(x) > 0$ for all $x \in \mathbb{R}$



$Y = g(X)$

Let dy and dx be small

$$\Pr(y < Y \le y + dy) \approx f_Y(y)\, dy$$

$$\Pr(y < Y \le y + dy) = \Pr(x < X \le x + dx)$$
$$\approx f_X(x)\, dx$$