11 APRIL 2012

Chromatic Functions

Q: In how many ways can you color a graph given n colors?

Def: $\chi_G(n)$, the chromatic function, is the # ~~num~~ of ways to color G given n colors.

Note $\chi_G = \min\{n \in \mathbb{N} \mid \chi_G(n) > 0\}$

what do you know about $\chi_G(n)$

→ If $G = G_1 \cup G_2$ are connected components, then

$$\chi_G(n) = \cancel{\chi_{G_1}} \; \chi_{G_1}(n) \cdot \chi_{G_2}(n)$$

→ If $G = K_m$, then $\chi_G(n) = \binom{n}{m} \cdot m! = n(n-1) \ldots (n-m+1)$ $\binom{m}{n}$?

as a function of n, grow like n^m

Deletion / Contraction

Ex $G =$  e want $\chi_G(n)$

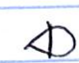
Let $G_{\hat{e}}$ (e has disappeared) be G without e

any coloring of G using n colors also colors $G_{\hat{e}}$. (Generally, it is easier to color things with less edges).

However, some coloring of $G_{\hat{e}}$ will not give a n-coloring of G. because the endpoints of e may receive the same color

Cook up a new graph G_e obtained from G by "contracting" the edge e.

We have

$G_{\hat{e}} =$  $G_e =$  ← becomes a parallel edge.

$\chi_{G_e}(n) = \chi_{\triangleleft}(n)$ (recall multiple edge makes no difference in color)

Deletion / Contraction Formula.

$$\underbrace{\chi_G(x)}_{\substack{\text{coloring of}\\ G\hat{e} \text{ where endpt.}\\ \text{of } e \text{ have diff}\\ \text{colors}}} = \underbrace{\chi_{G\hat{e}}(n)}_{\substack{\text{all coloring}\\ \text{of } G\hat{e}\\ \text{(counts too}\\ \text{much)}}} - \underbrace{\chi_{G_e}(n)}_{\substack{\text{coloring of}\\ G\hat{e} \text{ when}\\ \text{endpts are of}\\ \text{same color}}}$$

Helpful $G\hat{e}$ has fewer edge than G

G_e has fewer edge + vertices than G

example

$$\chi_G = \chi_{\Delta} - \chi_{\Delta}$$

let $\Delta \hat{}_e$ $\quad$ call recursive $\quad$ easy to compute

$$= (\chi_{\Delta\circ} - \chi_{\Delta}) - \chi_{\Delta}$$

$$= \chi_{\Delta\circ} - 2\chi_{\Delta}$$

Recall $G = G_1 \cup G_2$ $\quad$ ~~recurse~~

$$= (\chi_{\Delta} \cdot \chi_{\circ}) - 2\chi_{\Delta}$$

$$= \chi_{\Delta}(\chi_{\circ} - 2)$$

$$= \binom{n}{3}(3!)\left[\binom{n}{1}(1!) - 2\right]$$

$$= n(n-1)(n-2)(n-2)$$

$$= n(n-1)(n-2)^2 \qquad (*)$$

For a general G, deletion / contraction will express $\chi_G(n)$ in terms of sums / differences of products of $\chi_{complete(n)}$ of various sorts.

Note: $\chi_G(n)$ is a polynomial of degree V = # of vertices w/ int. coeff.

e.g.

$$= (\underbrace{\chi_{\Delta\hat{e}} \cdot \chi_{\circ}}_{} - 2\chi_{\Delta})$$

this is the source of degree 4 (from $(*)$)

$= \chi_{\Delta\circ}$ = all degree sourse

= same vert as G, less edges

= deletion part = (edge)

= V = # of vert of G

More Properties of $\chi_G(n)$

1. $\chi_G(n) = a_v n^v + a_{v-1} n^{v-1} + \cdots + a_1 n + a_0$

with $a_v = 1$, the coefficient alternates in system(?) and a_0 must be zero since if $n=0$ $\chi(0)$ must be zero.

Q: The complexity of $\chi_G(n)$

→ Deletion/Contraction algorithm yields a recursive algorithm.

↳ doubling of problem (while shortening pblm length)

execute in about $(1.62)^{v+e}$ steps.

∃ there exists algorithm that approximate $\chi_G(n)$ upto a factor of 2 in polynomial time

1) Computing $\chi_G(n)$ is #p-complete (

⇒ sort of np-complete except np = yes or no #p = counting

#NP Compt prblm

On "stupid" questions

$\chi_G(n)$ = # of coloring of G w/ n colors

IF $n \in \mathbb{N}$!

what if it's not? what is $\chi_n(-1)$?

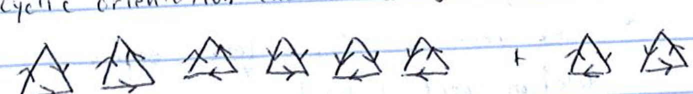
Richard Stanley

Given $G = \Delta$

- call orientation of G = process of attaching direct. to each edge of G
- call acyclic on orientation that does not contain oriented cycles.

ex. $G = \Delta$

acyclic orientation (no closed cycle) cycle

 $= 8 = 2^3$ ✓

no difference in color)

Stanley's Theorem

$|\chi_G(-1)| =$ # of acyclic orientation of G

= for any $-n$, $n \in \mathbb{N}$, some abstract concotion of acyclic orientation + coloring.

example

$|\chi_0(n)| = n(n-1)(n-2)$

$= (-1)(-2)(-3) = |-6| = 6$ ✓

Hyperplane Arrangement

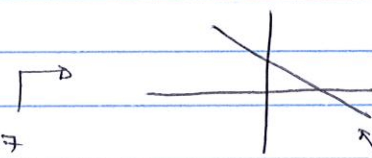
Q: Given a large sheet of paper and scissors, what is the maximum pieces you can cut w/ n straight cuts?

Ex: $n=0 \longrightarrow 1$

$n=1 \longrightarrow 2$

$n=2 \longrightarrow 4$

$n=3 \longrightarrow 8$? <u>NO</u> 7



no position of will penetrate all four quadrants.

Let $\chi_2(n) =$ the # of max pieces possible.

then, $\chi_2(n) \leq 2^n$

<u>Q</u> What if χ_3? Same idea, but think of cheese

$\chi_3(1) = 2$

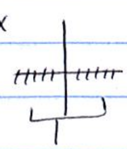
$\chi_3(2) = 4$

$\chi_3(3) = 8$ ✓

$\chi_4(4) = 16$? NO 15 0.

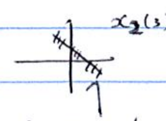
What is the recursion here?

ex



$\chi_2(2) \rightarrow$

two pieces of new line



$\chi_2(3)$

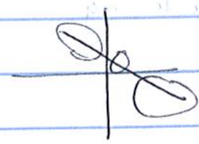
3 pieces of old line

the # of regions that is cut into 2 by the n^{th} line is equal to the # of segments into which the old line seperates the new

$\Rightarrow x_2(n) = x_2(n-1) + x_1(n-1)$

$\rightarrow$ chunks of cutting lines of previous graph by cutting the newly intr. line.

ex.



3 chuncks as a result of cuts

$x_2(n-1) = x_2(n) - x_1(n-1)$ ← this is really delete/contract...

to be cont...

no difference in color)