Independence of Events

Sometimes, knowledge of the occurrence of an event B does not affect the probability of an event A, i.e.

$$\Pr(A) = \Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

However, we would like a condotion for $\Pr(A) = \Pr(A \mid B)$ that does not break down when $\Pr(B) = 0$

Events A and B are <u>independent</u> if
$$\Pr(A \cap B) = \Pr(A)\Pr(B)$$

<u>Note</u>: $\Pr(A \cap B) = \Pr(A)\Pr(B)$
$$\Rightarrow \quad \Pr(A) = \Pr(A \mid B) \quad \text{and} \quad \Pr(B) = \Pr(B \mid A)$$

<u>Ex</u> Roll a die.

$A =$ "roll is divisible by 3" $= \{3, 6\}$

$B =$ "roll is ≥ 4" $= \{4, 5, 6\}$

$\Pr(A) = 1/3$, $\Pr(B) = 1/2$

$\Pr(A \cap B) = \Pr(\{6\}) = 1/6$

Thus
$$\Pr(A \cap B) = 1/6 = \Pr(A)\Pr(B)$$
and A, B are independent

Note: If $Pr(A), Pr(B) > 0$ and A, B are mutually exclusive then they cannot be independent.

Pf: Suppose A, B are mutually exclusive and independent.

mutually exclusive : $Pr(A \cap B) = 0$

independence : $Pr(A \cap B) = Pr(A) Pr(B)$

Combining these two properties we get that $Pr(A) Pr(B) = 0 \Rightarrow Pr(A) = 0$ or $Pr(B) = 0$ which is a contradiction.

Now consider events $A_1, \dots, A_n$

Events $A_1, \dots, A_n$ are independent if

$$Pr(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}) = Pr(A_{i_1}) Pr(A_{i_2}) \dots Pr(A_{i_k})$$

where $k = 2, \dots, n$ and $1 \le i_1 < i_2 < \dots < i_k \le n$

Note: For n events we need to verify $2^n - n - 1$ conditions to prove independence

Ex Three events A_1, A_2, A_3 are independent if

$$Pr(A_1 \cap A_2) = Pr(A_1)\,Pr(A_2)$$
$$Pr(A_1 \cap A_3) = Pr(A_1)\,Pr(A_3)$$
$$Pr(A_2 \cap A_3) = Pr(A_2)\,Pr(A_3)$$
$$Pr(A_1 \cap A_2 \cap A_3) = Pr(A_1)\,Pr(A_2)\,Pr(A_3)$$

In general, events of separate experiments are often assumed to be independent.

Ex (Ex 2.34 from text)
Suppose we flip a coin three times and we observe the sequence of ~~hed~~ heads and tails.

$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$
Assume that at each flip $Pr(\{H\}) = p$
~~(Pr({H})~~ $(Pr(\{T\}) = 1-p)$ and that the flips are independendent.

$$Pr(\{HHT\}) = Pr(\{H_1\})\,Pr(\{H_2\})\,Pr(\{T_3\})$$
$$= p^2(1-p)$$

Note: Unions, intersections, complements, differences on collections of independent events yield ~~independent~~ independent ~~are~~ events.

Ex: If A_1, A_2, A_3 independent, then

A_1 and $\bar{A}_2$ are independent

$A_1 - A_3$ and A_2 are independent

etc.

We will not prove this result for general events, but ~~ca~~ can give proofs for particular examples.

Ex Suppose A_1, A_2, A_3 are independent

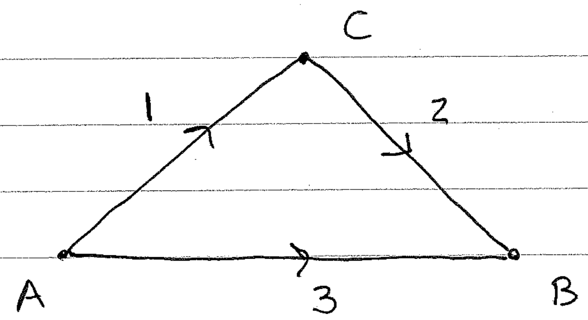
Can show $A_1, A_2 \cup A_3$ are independent

$$
\begin{aligned}
\Pr(A_1 \cap (A_2 \cup A_3)) &= \Pr((A_1 \cap A_2) \cup (A_1 \cap A_3)) \\
&= \Pr(A_1 \cap A_2) + \Pr(A_1 \cap A_3) \\
&\quad - \Pr(A_1 \cap A_2 \cap A_3) \\
&= \Pr(A_1)\Pr(A_2) + \Pr(A_1)\Pr(A_3) \\
&\quad - \Pr(A_1)\Pr(A_2 \cap A_3) \\
&= \Pr(A_1)\left[\Pr(A_2) + \Pr(A_3) - \Pr(A_2 \cap A_3)\right] \\
&= \Pr(A_1)\Pr(A_2 \cup A_3)
\end{aligned}
$$

$\therefore$ $A_1, A_2 \cup A_3$ are independent

<u>Ex</u> (pg. 25 Prof. Gelfand's Notes)

A communications network with three nodes A, B, C has three links 1, 2, 3



Links work (or fail) independently with probability 0.9. Possible to transmit from one node to another if there is a working path of links between them.

Let L_i = " Link i is working " $i = 1, 2, 3$

$Pr(L_i) = 0.9$

L_1, L_2, L_3 are independent.

a) Find the probability of being able to transmit from A to B.

We can transmit from A to B if L_1 <u>and</u> L_2 occur or if L_3 occurs

So we have

$Pr(A \to B)$

$= Pr((L_1 \cap L_2) \cup L_3)$

$= Pr(L_1 \cap L_2) + Pr(L_3) - Pr(L_1 \cap L_2 \cap L_3)$

$= Pr(L_1) Pr(L_2) + Pr(L_3) - Pr(L_1) Pr(L_2) Pr(L_3)$

since L_1, L_2, L_3 are independent.

$= (0.9)^2 + 0.9 - (0.9)^3 = 0.981$

This approach is much easier than finding the outcomes in $(L_1 \cap L_2) \cup L_3$ and using axiom (iii)

b) Find the probability we can't transmit from A to B given L_1 occurs (link 1 is working)

$Pr(A \not\to B \mid L_1)$

$= Pr(\bar{L}_2 \cap \bar{L}_3 \mid L_1)$

$= Pr(\bar{L}_2 \cap \bar{L}_3)$, since L_1, $\bar{L}_2 \cap \bar{L}_3$ independent

$= Pr(\bar{L}_2) Pr(\bar{L}_3)$, $\bar{L}_2$, $\bar{L}_3$ independent

$= (1 - Pr(L_2))(1 - Pr(L_3))$

$= 0.1^2 = 0.01$

<u>Ex</u> (pg. 30 of Prof. Gelfand's notes)

A pair of dice are rolled.
Let A be the event "get odd number on 1^{st} roll"
B be the event "get odd number on 2^{nd} roll"
C be the event "sum of rolls is odd"

Assume dice rolls are fair and independent.

a) Show A and C are independent

$$\Pr(A) = \Pr(B) = \Pr(C) = 1/2$$

$$\begin{aligned}\Pr(A \cap C) &= \Pr(A \cap \bar{B}) \\ &= \Pr(A)\Pr(\bar{B}) \\ &= 1/2 \cdot 1/2 \\ &= \Pr(A)\Pr(C)\end{aligned}$$

Similarly, B and C are independent
A and B are independent (given)

b) Show A, B, C are not independent.

$$\begin{aligned}\Pr(A \cap B \cap C) &= \Pr(\emptyset) \\ &= 0 \\ &\neq \tfrac{1}{2} \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} \\ &= \Pr(A)\Pr(B)\Pr(C)\end{aligned}$$

Hence, A, B, C are <u>not</u> independent