EE 301 **Exam 3** **4/8/13**

Academic Honesty Statement: I am aware of the course policies concerning academic honesty for Professor Krogmeier's section of ECE 301. Furthermore, I promise that the work I am submitting with this exam is my own work and that I have used no notes, materials, or other aids except as permitted by Professor Krogmeier.

Signature: ____________________

Name Printed: Solution

General Instructions:

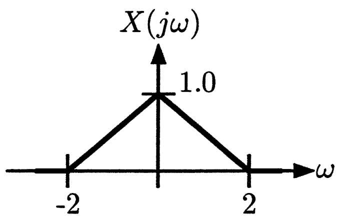
- **You have 60 minutes to complete the exam.**
- Write your name on every page of the exam.
- The exam is closed book and closed notes. Calculators are not allowed.
- Your work must be explained to receive full credit. All plots must be carefully drawn with axes labeled.
- Point values for each problem are as indicated. The exam totals 110 points of which at most 100 points can be counted.
- If you finish the exam during the first 50 minutes, you may turn it in and leave. During the last 10 minutes you must remain seated until we pick up exams from everyone.
- Problems labeled with **LO** indicate that the problem is used to determine student satisfaction of course learning objectives.

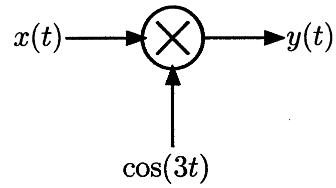
This exam is for Krogmeier's section of 301.

Do not open the exam until you are told to begin.

Name: ______________________

Problem 1. *Assorted Very Short Answer Questions.* [30 pts. total, **LO-iv**] For full credit you must have the correct answer *and* a brief explanation of it.

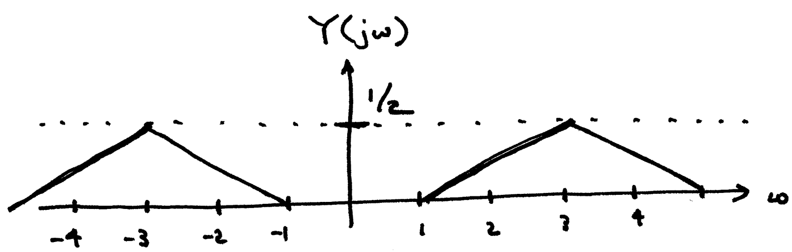




(a) [5 pts.] *Double-sideband, suppressed carrier amplitude modulation (AM).* Find and plot the spectrum $Y(j\omega)$ corresponding to $y(t)$ in the above block diagram.

$$y(t) = x(t)\cos(3t) \longleftrightarrow Y(j\omega) = \frac{1}{2\pi}\left\{X(j\omega) * \left[\pi\,\delta(\omega-3) + \pi\,\delta(\omega+3)\right]\right\}$$

$$= \frac{1}{2}X(j(\omega-3)) + \frac{1}{2}X(j(\omega+3))$$



(b) [5 pts.] *Comparison of required transmission bandwidths.* We have a large number of analog signal sources each of which has a baseband bandwidth of 1 KHz. We hope to use frequency division multiplexing to transmit these signals using a radio channel of bandwidth 1 MHz. Consider two different modulation schemes and answer the two questions below.

- If double sideband suppressed carrier AM is used, how many analog signal sources can be sent using the 1 MHz radio channel?

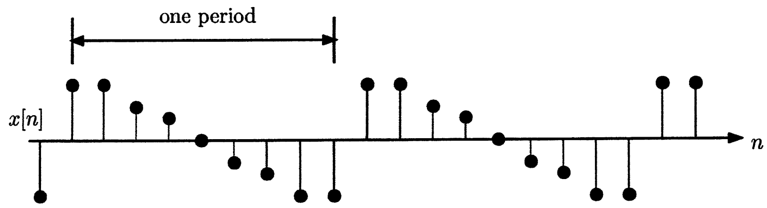
Transm. BW of DSB-AM is $2 \times \{\text{baseband BW}\} = 2\,\text{KHz}$ per analog source

$\therefore$ # sources supported $= \frac{1\,\text{MHz}}{2\,\text{KHz}} = 500$

- If single sideband AM is used, how many analog signal sources can be sent using the 1 MHz radio channel?

Trans. BW of SSB-AM is $1 \times \{\text{baseband BW}\} = 1\,\text{KHz}$ per analog source

$\therefore$ # sources supported $= \frac{1\,\text{MHz}}{1\,\text{KHz}} = 1000$



(c) [5 pts.] *Simple application of equations defining DTFS.* For the above periodic discrete time signal $x[n]$ with DTFS coefficients denoted by a_k, find the single DTFS coefficient a_0.

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j2\pi kn/N} \implies a_0 = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$$

But according to above sketch, the average value over one period is zero (signal amplitudes symm. wrt 0). Therefore

$$a_0 = 0$$

(d) [5 pts.] *Simple property of DTFT.* What is the highest frequency discrete-time signal and what is its DTFT?

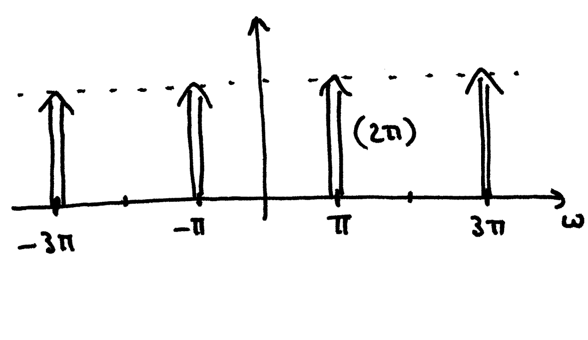
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \longleftrightarrow X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

There is no definitive correct answer to this question... just looking for a certain reasoning. Highest frequency corresponds to time signal changing most quickly in time. But since time is discrete the most it can do in the time of one sample is to change sign... i.e.,

$$x[n] = (-1)^n \quad n \in \mathbb{Z}$$

$$\updownarrow$$

$$X(e^{j\omega}) = 2\pi \sum_{\ell} \delta(\omega - \pi - 2\pi\ell)$$



Problem 1. (cont'd.) Name: ______________

(e) [5 pts.] *Simple application of equations defining the DTFT.* Let a discrete time signal $x[n]$ be defined by

$$x[n] = (0.8)^{n-1}u[n-1] + 1.25\delta[n] - (0.8)^{-n-1}u[-n-1].$$

Find the value of its DTFT evaluated at $\omega = 0$, i.e., find $X(e^{j0})$.

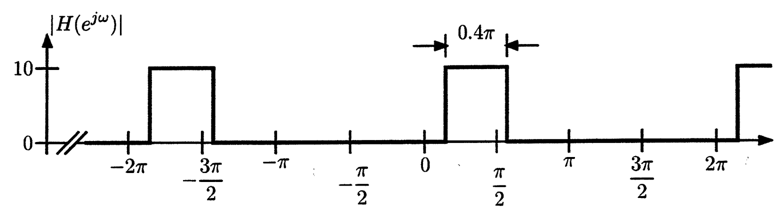
$$X(e^{j\omega}) = \sum_n x[n] e^{-j\omega n}$$

$$\therefore X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n]$$

Careful examination of given x[n] shows it has symm. st.

$$\sum_n x[n] = \underbrace{\sum_{n=1}^{\infty} (.8)^{n-1}}_{\text{these cancel}} + 1.25 - \underbrace{\sum_{n=-\infty}^{-1} (.8)^{-n-1}}_{\text{these cancel}}$$

$$\therefore X(e^{j0}) = \sum_n x[n] = 1.25$$



(f) [5 pts.] *Energy calculations in time and frequency domains.* Let $h[n] \leftrightarrow H(e^{j\omega})$ be a DTFT pair with the DTFT magnitude shown in the plot above. Find the energy in the time domain pulse:

$$\sum_{n=-\infty}^{\infty} |h[n]|^2.$$

From Parseval for DTFT

$$\sum_{n=-\infty}^{\infty} |h[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |H(e^{j\omega})|^2 \, d\omega$$

$$= \frac{1}{2\pi} \cdot 10^2 \cdot \frac{4}{10}\pi$$

$$= 20$$

Name: ____________________

Problem 2. *Solving LTI systems using the DTFT and its properties.* [40 pts. total, **LO-v**] This problem has three parts. Parts (b) and (c) are independent but both rely upon part (a).

(a) [10 pts.] An LTI discrete-time system is causal and characterized by a difference equation

$$y[n] - \frac{3}{4}y[n-2] = 2x[n].$$

Find the transfer function $H(e^{j\omega})$ of the system.

From the shifting property of DTFT and taking transform of both sides of the difference equation:

$$Y(e^{j\omega}) - \frac{3}{4}e^{-j2\omega}Y(e^{j\omega}) = 2X(e^{j\omega})$$

$$Y(e^{j\omega})\left[1 - \frac{3}{4}e^{-j2\omega}\right] = 2X(e^{j\omega})$$

$$\Longrightarrow \quad H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{2}{1 - \frac{3}{4}e^{-j2\omega}}$$

Problem 2. (cont'd.) Name: ____________

(b) [20 pts.] Assuming that the input is $x[n] = (1/2)^n u[n]$ find the output. You will likely need to use partial fraction expansion (summarized on the last page of the exam).

From the transform table: $X(e^{j\omega}) = \dfrac{1}{1-\frac{1}{2}e^{-j\omega}}$

$$\therefore \; Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \frac{2}{1-\frac{3}{4}e^{-j2\omega}} \, \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

Want inverse DTFT $y[n]$. Usual approach is to use PFE and the tables.

Let $v = e^{-j\omega}$. Then denom. of H is

$$1-\frac{3}{4}v^2 = \left(1-\frac{\sqrt{3}}{2}v\right)\left(1+\frac{\sqrt{3}}{2}v\right)$$

$$\therefore \; Y(v) = \frac{1}{\left(1-\frac{\sqrt{3}}{2}v\right)\left(1+\frac{\sqrt{3}}{2}v\right)\left(1-\frac{1}{2}v\right)} = \frac{A}{1-\frac{\sqrt{3}}{2}v} + \frac{B}{1+\frac{\sqrt{3}}{2}v} + \frac{C}{1-\frac{1}{2}v}$$

distinct roots.

$$A = \left(1-\frac{\sqrt{3}}{2}v\right)Y(v)\Big|_{v=2/\sqrt{3}} = \frac{1}{\left(1+\frac{\sqrt{3}}{2}\cdot\frac{2}{\sqrt{3}}\right)\left(1-\frac{1}{2}\cdot\frac{2}{\sqrt{3}}\right)} = \frac{1}{2\left(1-\frac{1}{\sqrt{3}}\right)} = \frac{1}{2\left(1-\frac{\sqrt{3}}{3}\right)}$$

$$= \frac{3}{2(3-\sqrt{3})}$$

$$B = \left(1+\frac{\sqrt{3}}{2}\right)Y(v)\Big|_{v=-2/\sqrt{3}} = \frac{1}{\left(1+\frac{\sqrt{3}}{2}\frac{2}{\sqrt{3}}\right)\left(1+\frac{1}{2}\frac{2}{\sqrt{3}}\right)} = \frac{1}{2(1+1/\sqrt{3})} = \frac{3}{2(3+\sqrt{3})}$$

$$C = \left(1-\frac{1}{2}v\right)Y(v)\Big|_{v=2} = \frac{1}{(1-\sqrt{3})(1+\sqrt{3})} = \frac{1}{1-3} = -\frac{1}{2}$$

$$\therefore \; Y(e^{j\omega}) = \frac{\frac{3}{2(3-\sqrt{3})}}{1-\frac{\sqrt{3}}{2}e^{-j\omega}} + \frac{\frac{3}{2(3+\sqrt{3})}}{1+\frac{\sqrt{3}}{2}e^{-j\omega}} - \frac{1/2}{1-\frac{1}{2}e^{-j\omega}}$$

$$\updownarrow$$

$$y[n] = \frac{3}{2(3-\sqrt{3})}\left(\frac{\sqrt{3}}{2}\right)^n u[n] + \frac{3}{2(3+\sqrt{3})}\left(-\frac{\sqrt{3}}{2}\right)^n u[n] - \frac{1}{2}\left(\frac{1}{2}\right)^n u[n]$$

Problem 2. (cont'd.) Name: ____________________

(c) [10 pts.] Assuming that the input is $x[n] = (-1)^n$ find the output. Note that in this case the input is periodic and has been applied to the system from infinitely far back in the past.

We still use $Y(e^{j\omega}) = H(e^{j\omega})\,X(e^{j\omega})$ but now

$$x[n] = (-1)^n \longleftrightarrow X(e^{j\omega}) = 2\pi \sum_{\ell=-\infty}^{\infty} \delta(\omega - \pi - 2\pi\ell)$$

$$\Longrightarrow \quad Y(e^{j\omega}) = \frac{2}{1 - \frac{3}{4} e^{-j2\omega}} \cdot 2\pi \sum_{\ell} \delta(\omega - \pi - 2\pi\ell)$$

$$= 2\pi \sum_{\ell} \underbrace{\left(\frac{2}{1 - \frac{3}{4} e^{-j2\omega}}\right)\Bigg|_{\omega = \pi + 2\pi\ell}} \cdot \delta(\omega - \pi - 2\pi\ell)$$

$$= \frac{2}{1 - \frac{3}{4} e^{-j2(\pi + 2\pi\ell)}} = \frac{2}{1 - \frac{3}{4}} = \frac{2}{1/4} = 8$$

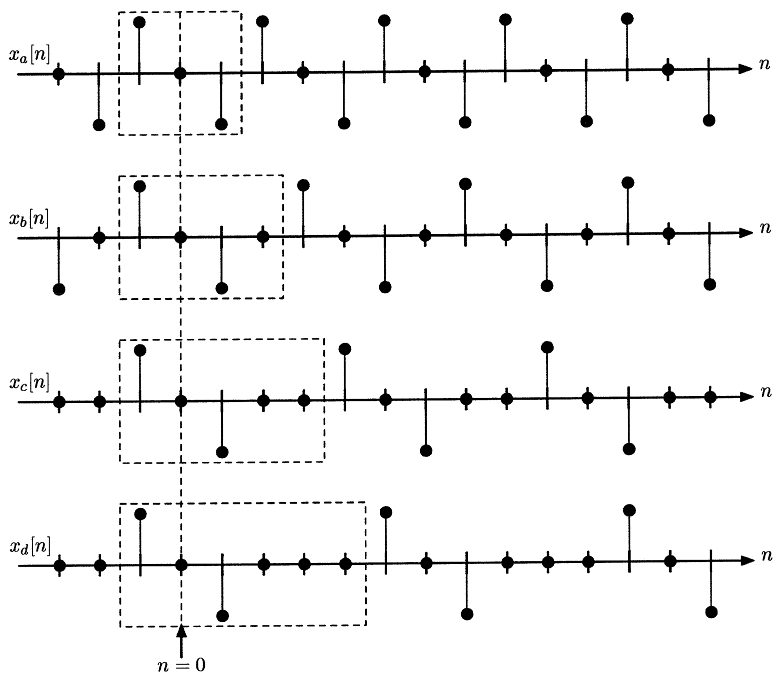
$$\therefore \quad Y(e^{j\omega}) = 2\pi \sum_{\ell=-\infty}^{\infty} 8 \cdot \delta(\omega - \pi - 2\pi\ell)$$

$$\updownarrow$$

$$y[n] = 8(-1)^n$$

note: gain of $H(e^{j\omega})$ is 8 at frequency $\omega = \pi$.

Name: ______________________



Problem 3. *Using the DTFT to simplify computation of DTFS.* [40 pts. total, **LO-iv**] Four periodic signals are given in the drawing above. All are created from an elementary signal $x[n] \stackrel{\text{def}}{=} \delta[n+1] - \delta[n-1]$ by repeating it periodically. For example:

$$x_a[n] = \sum_{k=-\infty}^{\infty} x[n-3k]$$

is periodic of period 3.

This problem has four parts.

(a) [9 pts.] Write each of the three remaining signals $x_b[n]$, $x_c[n]$, and $x_d[n]$ in an expansion in terms of the elementary signal $x[n]$ analogous to that given for $x_a[n]$. What is the period of each?

$x_b[n]$ is periodic of period 4; $x_b[n] = \sum_{k=-\infty}^{\infty} x[n-4k]$

$x_c[n]$ is periodic of period 5; $x_c[n] = \sum_{k=-\infty}^{\infty} x[n-5k]$

$x_d[n]$ " " " " 6; $x_c[n] = \sum_{k=-\infty}^{\infty} x[n-6k]$

(b) [9 pts.] Find and simplify the DTFT $X(e^{j\omega})$ of the elementary signal $x[n]$.

$$x[n] = \delta[n+1] - \delta[n-1]$$

$$X(e^{j\omega}) = \sum_n x[n] e^{-j\omega n} = \sum_n \delta[n+1] e^{-j\omega n} - \sum_n \delta[n-1] e^{-j\omega n}$$

$$= e^{+j\omega} - e^{-j\omega} = j2\sin(\omega)$$

(c) [12 pts.] Find and simplify the DTFS of the proper period for each of the four periodic signals defined in the plot. Denote these DTFS pairs by:

In general

$$x_*[n] = \sum_{k=-\infty}^{\infty} x[n-Mk]$$

$$x_a[n] \leftrightarrow X_{a,k}$$
$$x_b[n] \leftrightarrow X_{b,k}$$
$$x_c[n] \leftrightarrow X_{c,k}$$
$$x_d[n] \leftrightarrow X_{d,k}.$$

Hint: All of these can be written in terms of samples of the elementary DTFT $X(e^{j\omega})$.

The DTFS of $x_*[n] \longleftrightarrow X_{*,k}$ is given by

$$X_{*,k} = \frac{1}{M} \sum_{\substack{\text{any period} \\ \text{of length} \\ M}} x_*[n] e^{-jk(2\pi/M)n} = \frac{1}{M} \sum_{n=-1}^{M-2} x[n] e^{-jk(2\pi/M)n}$$

$$= \frac{1}{M} X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{M}} = \frac{j2}{M} \sin\left(\frac{2\pi k}{M}\right)$$

__For $x_a[n]$__ $\quad X_{a,k} = \frac{j2}{3} \sin\left(\frac{2\pi k}{3}\right)$

__For $x_b[n]$__ $\quad X_{b,k} = \frac{j2}{4} \sin\left(\frac{2\pi k}{4}\right)$

__For $x_c[n]$__ $\quad X_{c,k} = \frac{j2}{5} \sin\left(\frac{2\pi k}{5}\right)$

__For $x_d[n]$__ $\quad X_{d,k} = \frac{j2}{6} \sin\left(\frac{2\pi k}{6}\right)$

Problem 3. (cont'd.) Name: ______________

(d) [10 pts.] Now let $y[n] \leftrightarrow Y(e^{j\omega})$ be an arbitrary DTFT pair. For any positive integer M create a periodic sequence (of period M) by sampling the DTFT:

$$\tilde{Y}_k = \frac{1}{M} Y(e^{j\omega})\Big|_{\omega = 2\pi k/M}, \quad \text{for } k \in \mathcal{Z}.$$

Let $\tilde{y}[n]$ be the inverse M-point DTFS of $\tilde{Y}_k$. Derive a formula expressing $\tilde{y}[n]$ in terms of $y[n]$.

$$\tilde{y}[n] = \sum_{k=0}^{M-1} \frac{1}{M} Y(e^{j2\pi k/M})\, e^{j2\pi k n/M} = \frac{1}{M} \sum_{k=0}^{M-1} \sum_{\ell=-\infty}^{\infty} y[\ell]\, \underbrace{e^{-j2\pi k \ell/M}\, e^{j2\pi k n/M}}_{e^{j2\pi k(n-\ell)/M}}$$

Make c.o.v. in sum. Let $p = n - \ell$. Then

$$\tilde{y}[n] = \frac{1}{M} \sum_{k=0}^{M-1} \sum_{p=-\infty}^{\infty} y[n-p]\, e^{j2\pi k p/M}$$

$$= \frac{1}{M} \sum_{p=-\infty}^{\infty} y[n-p] \boxed{\sum_{k=0}^{M-1} e^{j2\pi k p/M}} \longrightarrow = \begin{cases} M & \text{if } p = qM \\ 0 & \text{else} \end{cases}$$

$$= \sum_{\substack{p=-\infty \\ p = qM}}^{\infty} y[n-p] = \sum_{q=-\infty}^{\infty} y[n-qM]$$