MA544 Qual Review Problems 5 Bridges Summer 08

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1. Let $f_n \in L^1(0,1)$ such that $f_n \to f$ in L^1. Show there exists $M \in \mathbb{N}$ such that
$$\limsup_{n\to\infty} \int_{\{|f_n|>M\}} |f_n| = 0.$$

2. Let $(X, \mathcal{M}, \mu)$ be a measure space, f_n and f are μ-measureable functions on X.
   (a) Prove $f_n \to f$ in measure implies there is a subsequence $\{f_{n_k}\}_k$ converging to f a.e.
   (b) Suppose now that $f_n \to f$ a.e. Prove or disprove that f_n converges to f in measure, under the following hypotheses:
      i. $\mu(X) < \infty$.
      ii. μ is sigma-finite
      iii. No additional assumptions on our space.

3. Let $(X, \mathcal{M}, \mu)$ be a sigma-finite measure space and let $f \in L^p(\mu), 1 \leq p < \infty$. Show
$$\int_X |f|^p d\mu = p \int_{-\infty}^{\infty} \lambda^{p-1} \mu(\{|f| > \lambda\}) d\lambda.$$
Where did you use sigma-finiteness?

4. Two real-valued functions f and g defined on $[0,1]$ are said to be comonotone if:
$$(f(x) - f(y))\,(g(x) - g(y)) \geq 0$$
for all $x, y \in [0,1]$. Suppose the two functions are Lebesgue measurable. Prove that:
$$\left(\int_0^1 f(t)dt\right)\left(\int_0^1 g(t)dt\right) \leq \int_0^1 f(t)g(t)dt.$$

5. Let $f \in L^2(\mathbb{R})$ and let $f_0(x) = xf(x)$. Show that $\|f\|_1 \leq (8\|f\|_2 \; \|f_0\|_2)^{1/2}$. (Hint: consider $\{|x| > a\}$ and $\{|x| \leq a\}$.)

6. Let $g \in L^p(\mathbb{R}), 1 \leq p < \infty$ and $f(x) = e^{-|x|}$. Show where you use Fubini's and Tonelli's Theorems respectively to prove

$$\int_{-\infty}^{\infty} (f * g)^p(x) dx = \frac{2}{p} \int_{-\infty}^{\infty} g(x)^p dx.$$

7. Let $\mathcal{A}$ be the σ-algebra in $\mathbb{R}^2$ generated by the family of sets $\{(x, y) : x \geq r_1, y \in [r_2, r_3)\}$, where $r_j \in \mathbb{Q}$ for $j = 1, 2, 3.$. Prove or disprove $\mathcal{A}$ is the Borel σ-algebra on $\mathbb{R}^2$.

8. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Let $f_n \to f$ in $L^p, 1 \leq p < \infty$. Fix $\epsilon > 0$ and show there exists $\delta = \delta(\epsilon) > 0$ such that $\forall A \in \mathcal{M}$ with $\mu A < \delta$, we have $\int_A |f_n| < \epsilon$.

9. (Test 2-2) Let $f \in L^1([0, 1])$. Prove that the function

$$G(t) = \int_0^1 \cos(tf(x)) dx$$

is differentiable with respect to t.