1. Let $(X, \mathcal{S}, \mu)$ be a measure space and let $\{f_n\}$ be a sequence of non-negative, $\mathcal{S}$-measurable functions on X that converges a.e. to 0. Prove that if there exists a real number M such that
$$\int \sup\{f_1, \ldots, f_n\} d\mu \leq M, \ \forall n \in \mathbb{N}$$
then $\lim_n \int f_n d\mu = 0$.

2. (P8-2) Let $G_1 \supset G_2 \supset \cdots$ be a sequence of open sets in $I = [0, 1]$ and let
$$f(x) = \sum_{n=1}^{\infty} m([0, x] \cap G_n).$$
   (a) Show that $f \in AC(I)$ if and only if $\sum_{n=1}^{\infty} m(G_n) < \infty$.
   (b) Show that f is Lipschitz in I if and only if there exists $N \in \mathbb{N}$ such that $G_n = \emptyset, n \geq N$.

3. Let f be a Lebesgue measurable function on $\mathbb{R}$ and let f^* be the Hardy-Littlewood maximal function, i.e.
$$f^*(x) = \sup \frac{\int_Q |f|}{|Q|},$$
where the supremum is over all cubes Q centered at x.
   (a) Disprove or provide an example: $\exists f$ as above with $|\{f^* < f\}| > 0$
   (b) Suppose $f \neq 0$ and show f^* is not in L^1.

4. Evaluate with proof:
   (a) $\lim \int_0^\pi \sin^n(x) dx$
   (b) $\lim \int_0^\pi 2^n \sin^n(x) dx$

5. Let f_n be strictly positive Lebesgue measurable functions on $\mathbb{R}$ which converge pointwise to zero. Pick a number $\alpha > 0$ and prove there is a Lebesgue measurable set A with $|A| = \alpha$, and a subsequence $\{f_{n_k}\}_k$ such that $f_{n_k} > f_{n_{k+1}}$ on A.